1 Some geodesics

Using the variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion, for

a) the *Schwarzschild* metric

$$g_m = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(you might wish to do the calculation for a metric of the form

$$e^{2f(r)}dt^2 - e^{-2f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) ,$$

and specialize the result to Schwarzschild at the end, as many other important metrics are of this form, so the result might be useful later). Show that a geodesic initially tangent to the equatorial plane always remains in it.

b) The following *pp-wave* metric

$$g = -dx^2 - dy^2 + 2du\, dv - H(u, x)du^2 .$$

c) The "post-Newtonian" metric

$$g_{00} = 1 - \frac{2GM}{r} , \quad g_{0i} = 0 , \quad g_{ij} = -\left(1 - \frac{2GM}{r}\right)\delta_{ij} ,$$

with $i, j \in \{1, 2, 3\}$. (This is the Newtonian approximation, for $GM/r \ll 1$, of the metric tensor of a spherically symmetric body of mass M.) In your calculation neglect all terms quadratic in GM.