1 Newtonian warm-up

An elevator shaft is drilled diametrically through an ideally homogeneous spherical earth of radius R and density ρ. An elevator is dropped into this shaft from rest at the surface. Two point particles A and A', are initially at rest in the elevator, one at the center of the ceiling and one at the center of the floor, distance d apart. (i) Show that the whole cabin executes simple harmonic motion with respect to the shaft, the period of which you will determine. (ii) Find the relative acceleration between the particles in the frame defined by the center of mass of the elevator. (This relative acceleration is often referred to as "tidal gravitational force"). *[Hint: Use Gauss' theorem, equating influx of the gravitational field through a given surface to* $4\pi G$ *the enclosed mass.]*

2 Summation convention

For each of the following, either write out the equation with the summation signs included explicitly or explain why the equation is ambiguous or does not make sense. Provide a possible correct version, or versions, of the wrong or incoherent equations.

(i) $x^a = L^a{}_b M^{bc} \hat{x}_c$	(v) $x^a = L^a{}_b \hat{x}^b + M^{ab} \hat{x}^b$
(ii) $x^a = L^b{}_c M^c{}_d \hat{x}^d$	(vi) $x^a = L^a{}_b \hat{x}^b + M^a{}_c \hat{x}^c$
(iii) $\delta^a_b = \delta^a_c \delta^c_d$	(vii) $x^a = L^a{}_c \hat{x}^c + M^b{}_c \hat{x}^c$
(iv) $\delta^a_b = \delta^a_b \delta^c_c$	(viii) $x^a = L^a{}_c \hat{x}^c + \sum_c M^{ac} \hat{x}^c$

3 Changes of coordinates

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in each of the following coordinate systems.

(i) $\tilde{x}^0 = t - z$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$

(ii) $\tilde{x}^0 = t + r$, $\tilde{x}^1 = t - r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = \phi$

(iii) $\tilde{x}^0 = \tau$, $\tilde{x}^1 = \phi$, $\tilde{x}^2 = y$, $\tilde{x}^3 = z$, where, in the first case, r, θ are plane polar coordinates in the x, y plane; in the second, r, θ, ϕ are spherical polar coordinates; and, in the third, τ, ϕ are 'Rindler coordinates', defined by $t = \tau \cosh \phi$, $x = \tau \sinh \phi$. In each case, state which region of Minkowski space the coordinate system covers. *[Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ *and substitute, for example,* $\mathrm{d}x = \cos\theta \mathrm{d}r - r \sin\theta \mathrm{d}\theta$*, and so on. Of course you should justify that this is legitimate.]*