Eigenwerte, Eigenvektoren, Diagonalisierung

Vorlesungsnotizen zu

Mathematische Methoden der Physik I

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1 Basics

vector space V of dimension $\dim V = n$ over the field $\mathbb{R}$ (or $\mathbb{C}$)

basis

$$\{b_1, b_2, \dots, b_n\}$$

every vector $v \in V$ is represented as a unique linear combination of the basis vectors

$$v = v_1 b_1 + v_2 b_2 + \cdots + v_n b_n = \sum_{i=1}^{n} v_i b_i$$

different bases lead to different components

basis $\{\hat{b}_1, \dots, \hat{b}_n\}$ and basis $\{\check{b}_1, \dots, \check{b}_n\}$

$$v = \sum_{i=1}^{n} \hat{v}_i \hat{b}_i = \sum_{i=1}^{n} \check{v}_i \check{b}_i$$

the components $(\hat{v}_i)_{i=1,\dots,n}$ and $(\check{v}_i)_{i=1,\dots,n}$ are completely different but represent one and the same vector (w.r.t. two different bases)

once a basis has been chosen (or if it is clear which basis is used), it is customary to write the components of a vector v (w.r.t. that basis) as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

one speaks of a 'column vector'

an endomorphism is a linear map A of the vector space V onto itself, i.e.,

$$A : V \to V$$

such a linear map takes vectors $v \in V$ and maps these to $A(v) \in V$. it is customary to write Av instead of $A(v)$, because of the linearity of A.

once a basis has been chosen (or if it is clear which basis is used), the linear map is represented by a matrix, which is usually denoted by the same letter

the map A is linear, hence

$$Av = A\big(\sum_{j=1}^n v_j b_j\big) = \sum_{j=1}^n v_j\, Ab_j$$

now, Ab_j is a vector in V and can thus be domposed w.r.t. basis

$$Ab_j = \sum_{i=1}^n A_{ij} b_i$$

it follows that

$$Av = \sum_{j=1}^n v_j\, Ab_j = \sum_{j=1}^n v_j\, \sum_{i=1}^n A_{ij} b_i = \sum_{i=1}^n \big(\sum_{j=1}^n A_{ij} v_j\big)\, b_i$$

therefore the $i = 1, \ldots, n$ components of the transformed vector are

$$\sum_{j=1}^n A_{ij} v_j$$

collecting the components A_{ij} into a matrix

$$A = \big(A_{ij}\big)_{i,j=1,\ldots,n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

we obtain the transformed vector Av in its column vector representation through matrix multiplication

$$Av = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

it is a common source of confusion to denote the linear map A and the matrix representation of A (w.r.t. a basis) by the same letter. always keep in mind that while the linear map is an abstract (and fixed) entity, its matrix representation is not at all fixed but depends on the basis we choose. different basis: same map but different matrix.

2 Eigenvalues and eigenvectors

a vector $v \neq \vec{o}$ is an eigenvector of a linear map A if it is merely stretched or compressed by the map A, i.e., if there is a number λ such that

$$Av = \lambda v$$

eigenvectors are the linear map's pampered children. while the linear map might have some nasty effect on a general vector (rotate, reflect,...), the map is rather kind to an eigenvector: the effect of the map on an eigenvector is to simply multiply it by a number.

the number λ is called the eigenvalue that is associated with the eigenvector v.

EXAMPLE

Consider the vector space $\mathbb{R}^2$ (with the standard basis) and the linear map represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} .$$

The vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

is an eigenvector, since

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The associated eigenvalue is 2.

Example

The set of $\mathcal{C}^\infty$ functions $x \mapsto f(x)$ forms a vector space, and

$$\frac{d}{dx}$$

is a linear map. The function

$$e^{-x}$$

is an eigenvector ('eigenfunction') since

$$\frac{d}{dx}\, e^{-x} = -e^{-x} \,.$$

The associated eigenvalue is (-1). (Since the vector space of $\mathcal{C}^\infty$ functions is not finite dimensional, we cannot operate with matrices.)

v is an eigenvector of A if and only if there exists λ such that

$$Av = \lambda v \quad \Leftrightarrow \quad (A - \lambda \mathbb{1})v = \vec{o}$$

reinterpreting this equation we see that the set of eigenvalues of A is the set of numbers λ such that

$$(A - \lambda \mathbb{1})v = \vec{o}$$

possesses a non-trivial solution v.

this implies a number of equivalent statements.

$$\begin{aligned}
\lambda \text{ is eigenvalue of } A \quad & \Leftrightarrow \quad (A - \lambda \mathbb{1})v = \vec{o} \text{ has non-trivial solutions } v \\
& \Leftrightarrow \quad \ker(A - \lambda \mathbb{1}) \neq \{\vec{o}\} \\
& \Leftrightarrow \quad (A - \lambda \mathbb{1}) \text{ is not invertible} \\
& \Leftrightarrow \quad \det(A - \lambda \mathbb{1}) = 0
\end{aligned}$$

consequently, to obtain the set of eigenvalues of A we must solve the equation

$$\det(A - \lambda \mathbb{1}) = \begin{vmatrix} A_{11} - \lambda & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} - \lambda & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} - \lambda \end{vmatrix} = 0$$

the expression $\det(A - \lambda\mathbb{1})$ is called the characteristic polynomial of A. it is a polynomial of degree n,

$$\det(A - \lambda\mathbb{1}) = (-1)^n \Big(\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \Big)$$

the coefficients $(c_i)_{i=0,\ldots,n-1}$ are determined by the components $(A_{ij})_{i,j=1,\ldots,n}$ of A. the eigenvalues of A are the zeros of the characteristic polynomial, i.e., the solutions of the equation

$$\det(A - \lambda\mathbb{1}) = (-1)^n \Big(\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \Big) = 0$$

how many eigenvalues does a linear map A possess? it is crucial to distinguish real vector spaces and complex vector spaces.

consider a vector space over the field $\boxed{\mathbb{R}}$. then the linear map A is represented by a real matrix (i.e., a matrix whose entries are real) and the coefficients of the characteristic polynomial are real. the eigenvalues of A are the (real!) zeros of the characteristic polynomial. the characteristic polynomial is a polynomial of degree n.

therefore, if n is even, the number of eigenvalues of the map A can be anything between 0 and n; if n is odd, the number of eigenvalues of the map A can be anything between 1 and n.

EXAMPLE

Consider the vector space $\mathbb{R}^2$ and the linear maps represented by the matrices

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomials are

$$\begin{aligned}
\det(A_0 - \lambda \mathbb{1}) &= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0\,, \\
\det(A_1 - \lambda \mathbb{1}) &= \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 = 0\,, \\
\det(A_2 - \lambda \mathbb{1}) &= \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&A_0 \text{ has no eigenvalue} && \nexists\,\lambda\,, \\
&A_1 \text{ has one eigenvalue} && \lambda = 1\,, \\
&A_2 \text{ has two eigenvalues} && \lambda_1 = 1,\ \lambda_2 = -1\,.
\end{aligned}$$

We see that real (2×2) matrices can have 0, 1, or 2 eigenvalues.

consider a vector space over the field $\boxed{\mathbb{C}}$. then the linear map A is represented by a complex matrix (i.e., a matrix whose entries are complex) and the coefficients of the characteristic polynomial are complex. note that this does not necessarily mean that a complex matrix features an i somewhere (but it could). e.g., both

$$\begin{pmatrix} 1 & 4+i & -i \\ 0 & 0 & 1 \\ 1-i & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & 0 & -5 \end{pmatrix}$$

are complex matrices. (since $\mathbb{R} \subset \mathbb{C}$ the real numbers are automatically complex numbers.)

the eigenvalues of A are the (complex) zeros of the characteristic polynomial. since this is a polynomial of degree n, the number of eigenvalues of the map A can be anything between 1 and n. (the fundamental theorem of algebra states that every polynomial has at least one (complex) zero.)

EXAMPLE

Consider the vector space $\mathbb{C}^2$ and the linear maps represented by the matrices

$$B_1 = \begin{pmatrix} 1+i & 3-i \\ 0 & 1+i \end{pmatrix} , \quad B_2 = \begin{pmatrix} i & -2 \\ 1 & 0 \end{pmatrix} ,$$

The characteristic polynomials are

$$\det(B_1 - \lambda \mathbb{1}) = \begin{vmatrix} 1+i-\lambda & 3-i \\ 0 & 1+i-\lambda \end{vmatrix} = (1+i-\lambda)^2$$

$$= \lambda^2 - 2(1+i)\lambda + (1+i)^2 = 0 ,$$

$$\det(B_2 - \lambda \mathbb{1}) = \begin{vmatrix} i-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = -(i-\lambda)\lambda + 2 = \lambda^2 - i\lambda + 2 = 0 .$$

Therefore,

B_1 has one eigenvalue $\lambda = 1 + i$,
B_2 has two eigenvalues $\lambda_1 = 2i,\ \lambda_2 = -i$.

We see that complex (2×2) matrices can have 1 or 2 eigenvalues.

EXAMPLE

Consider the vector space $\mathbb{C}^2$ and the linear maps represented by the matrices

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomials are

$$\det(A_0 - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

$$\det(A_1 - \lambda \mathbb{1}) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 = 0,$$

$$\det(A_2 - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0.$$

Therefore,

$$\begin{aligned} &A_0 \text{ has two eigenvalues } \lambda_1 = i,\ \lambda_2 = -i, \\ &A_1 \text{ has one eigenvalue } \ \lambda = 1, \\ &A_2 \text{ has two eigenvalues } \lambda_1 = 1,\ \lambda_2 = -1. \end{aligned}$$

We see that complex (2×2) matrices can have 1 or 2 eigenvalues.

like every polynomial the characteristic polynomial can be factorized by using its roots. suppose that there are r roots (eigenvalues) $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$. (we know that $1 \leq r \leq n$). then

$$\begin{aligned} \det(A - \lambda \mathbb{1}) &= (-1)^n \Big(\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0\Big) \\ &= (-1)^n \big(\lambda - \lambda_1\big)^{m_1} \big(\lambda - \lambda_2\big)^{m_2} \cdots \big(\lambda - \lambda_r\big)^{m_r} \end{aligned}$$

where

$$m_1 + m_2 + \cdots m_r = n.$$

the integer numbers $m_1, \dots, m_r$ are the multiplicities of the roots $\lambda_1, \dots, \lambda_r$; in our present context we say that $m_1, m_2, \dots, m_r$ are the *algebraic multiplicities* of the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$. (note that $\sum_{i=1}^n m_i = n$ is a specialty of the complex numbers; in the case of a real vector space, the sum of the algebraic multiplicities can be less than n.)

consider a linear map A of the vector space V onto itself. the field can be $\boxed{\mathbb{R}}$ or $\boxed{\mathbb{C}}$. suppose that λ is an eigenvalue of A. the associated eigenvectors are obtained by solving the equation

$$Av = \lambda v \quad \Leftrightarrow \quad (A - \lambda \mathbb{1})v = \vec{o}\,.$$

this is a system of linear equations. the existence of non-trivial solutions v is guaranteed by the fact that $\det(A - \lambda\mathbb{1}) = 0$ (since λ is an eigenvalue).

applying the theory of systems of linear equations we see that the solutions of $(A - \lambda\mathbb{1})v = \vec{o}$ form a (non-trivial) linear subspace E_λ in V,

$$E_\lambda = \ker(A - \lambda\mathbb{1})\,.$$

each vector $v \in E_\lambda$ satisfies the equation $(A - \lambda\mathbb{1})v = \vec{o}$ and is thus an eigenvector of A with eigenvalue λ. we call the space E_λ the eigenspace of the map A associated with the eigenvalue λ.

Example

Consider the vector space $\mathbb{C}^3$ and the linear map represented by the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{1}{2} \end{pmatrix} .$$

Computing the eigenvalues we obtain

$$\left|A - \lambda \mathbb{1}\right| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\frac{1}{2}-\lambda & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{1}{2}-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} -\frac{1}{2}-\lambda & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2}-\lambda \end{vmatrix}$$

$$= (2-\lambda)\Big(\big(\lambda + \tfrac{1}{2}\big)^2 - \tfrac{25}{4}\Big) = (2-\lambda)\big(\lambda^2 + \lambda - 6\big) = 0 .$$

Accordingly, one eigenvalue is 2; the remaining eigenvalue(s) are obtained by solving the quadratic equation $\lambda^2 + \lambda - 6 = 0$,

$$\frac{-1 \pm \sqrt{1+24}}{2} = \{-3, 2\} .$$

The eigenvalue 2 appears again and the number (-3) is one more eigenvalue. Accordingly, in the present example the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -3$. The algebraic multiplicities of the eigenvalues are $m_1 = 2$ and $m_2 = 1$, because the characteristic polynomial reads

$$-(\lambda-2)\big(\lambda^2+\lambda-6\big) = -(\lambda-2)(\lambda-2)(\lambda+3) = -(\lambda-2)^2(\lambda+3) .$$

Let us compute the eigenspace E_1 (the set of eigenvectors) associated with $\lambda_1 = 2$.

$$(A - \underbrace{\lambda_1}_{2} \mathbb{1}\,)\, v = \vec{o} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{5}{2} & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The only equation we get is $-\frac{5}{2}\, v_2 + \frac{5}{2}\, v_3 = 0$; hence the solution is

$$E_1 = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\Big|\; v_2 = v_3 \right\} .$$

To be continued...

EXAMPLE

...and now the continuation. The eigenspace E_1 is a two-dimensional subspace of V; it is the space of eigenvectors of A w.r.t. the eigenvalue $\lambda_1 = 2$. Every vector in E_1 is an eigenvector of A w.r.t. λ_1; examples are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

Since E_1 is two-dimensional, it is spanned by any two (linearly independent) vectors in E_1; for instance we can write

$$E_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle,$$

where $\langle \cdot \rangle$ denotes the linear span. (Recall that the linear span $\langle v_1, \dots, v_n \rangle$ is defined as $\{c_1 v_1 + \dots + c_n v_n\}$ with constants $c_1, \dots, c_n$.)

Analogously, we compute the eigenspace E_2 associated with $\lambda_2 = -3$.

$$(A - \underbrace{\lambda_2}_{-3} \mathbb{1}) v = \vec{o} \quad \Leftrightarrow \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & \frac{5}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain two independent equations: $5v_1 = 0$ and $\frac{5}{2} v_2 + \frac{5}{2} v_3 = 0$; hence the solution is

$$E_2 = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\middle|\; v_1 = 0 \,\wedge\, v_2 = -v_3 \right\}.$$

This eigenspace is spanned by one vector and thus one-dimensional; we write

$$E_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle.$$

Every vector in E_2 is an eigenvector associated with the eigenvalue $\lambda_2 = -3$. *To be continued...*

EXAMPLE

... and now the conclusion. Let us summarize. The linear map represented by the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{1}{2} \end{pmatrix} .$$

possesses two eigenvectors: $\lambda_1 = 2$ and $\lambda_2 = -3$. The associated spaces of eigenvectors (eigenspaces) E_1 and E_2 are

$$E_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \qquad E_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle .$$

The algebraic multiplicity of $\lambda_1 = 2$ is $m_1 = 2$; the algebraic multiplicity of $\lambda_2 = -3$ is $m_2 = 1$.

Let us define the *geometric multiplicity* d_λ of an eigenvalue λ as the dimension of the associated eigenspace E_λ. In our example we get $d_1 = \dim E_1 = 2$ and $d_2 = \dim E_2 = 1$. Comparing the algebraic multiplicities with the geometric multiplicities we see that

$$d_1 = m_1 = 2 , \qquad d_2 = m_2 = 1 .$$

An obvious question to ask is whether this statement generalizes: Does the geometric multiplicity always coincide with the algebraic multiplicity? Unfortunately, as we will see in the subsequent example, the answer is no.

Example

Consider the vector space $\mathbb{C}^3$ and the linear map represented by the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

To compute the eigenvalues we calculate the characteristic polynomial; we obtain

$$\begin{aligned} \left|A - \lambda \mathbb{1}\right| = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} &= (-1-\lambda) \begin{vmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (-1-\lambda)\big(-\lambda(2-\lambda)+1\big) = -(\lambda+1)\big(\lambda^2 - 2\lambda + 1\big) \\ &= -(\lambda+1)(\lambda-1)^2 = 0 . \end{aligned}$$

Accordingly, there exist two eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 1$; the algebraic multiplicities are $m_1 = 1$ and $m_2 = 2$.

Computing the eigenvectors (eigenspace) associated with $\lambda_1 = -1$ we obtain

$$\big(A - \lambda_1 \mathbb{1}\big)\, v = \big(A + \mathbb{1}\big)\, v = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

hence $v_1 + v_2 = 0$, $v_1 + 3v_2$ and therefore $v_1 = 0$ and $v_2 = 0$. Accordingly,

$$E_1 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle .$$

Analogously, we compute the eigenvectors (eigenspace) associated with $\lambda_2 = 1$. We obtain

$$\big(A - \lambda_1 \mathbb{1}\big)\, v = \big(A - \mathbb{1}\big)\, v = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

hence $v_1 + v_2 = 0$ and $v_3 = 0$. *To be continued. . .*

Example

... and now the conclusion. Accordingly,

$$E_2 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle .$$

We see that the geometric multiplicities are

$$d_1 = \dim E_1 = 1 , \qquad d_2 = \dim E_2 = 1 .$$

In particular, we conclude that the geometric multiplicity of the eigenvalue $\lambda_2 = 1$ is less than its algebraic multiplicity.

$$d_1 = m_1 = 1 , \qquad d_2 = 1 \; < \; m_2 = 2 .$$

This statement is true in general. The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

consider a linear map A of V onto itself. let λ be an eigenvalue of A with algebraic multiplicity m_λ; let $E_\lambda = \ker(A - \lambda \mathbb{1})$ denote the space of eigenvectors (eigenspace) associated with λ. we define the *geometric multiplicity* d_λ of λ as the dimension of the associated eigenspace E_λ,

$$d_\lambda = \dim E_\lambda .$$

then there is the following important statement:

$$1 \leq d_\lambda \leq m_\lambda ;$$

in particular, the geometric multiplicity of an eigenvalue λ is less than or equal to its algebraic multiplicity.

suppose the linear map A possesses the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ with geometric multiplicities $d_1, d_2, \ldots, d_r$, i.e., the associated eigenspaces $E_1, E_2, \ldots, E_r$ satisfy $\dim E_i = d_i \; \forall i = 1, \ldots, r$. so, how many linearly independent eigenvectors does the map A have? the answer is

$$d_1 + d_2 + \cdots + d_r .$$

since $d_i \leq m_i$ and $m_1 + \cdots + m_r = n$ (or $\leq n$ in the case of real vector spaces) we obtain

$$d_1 + d_2 + \cdots + d_r \leq n .$$

this is a non-trivial statement, which follows essentially from the fact that eigenvectors associated with different eigenvalues are always linearly independent. (we omit the proof.)

an alternative way of stating the above is

$$E_1 + \cdots + E_r = E_1 \oplus \cdots \oplus E_r \quad (\subseteq V)\,,$$

or

$$\dim\big(E_1 \oplus \cdots \oplus E_r\big) = \dim E_1 + \cdots + \dim E_r = d_1 + \cdots + d_r \quad (\leq n)\,.$$

every eigenspace E_i adds d_i linearly independent eigenvectors to the set of eigenvectors. (we never have to worry that we get a linearly dependent one.)

we conclude this section with some useful remarks and observations.

triangular matrices

consider an (upper or lower) triangular matrix, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ 0 & A_{22} & A_{23} & \cdots & A_{2n} \\ 0 & 0 & A_{33} & \cdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{nn} \end{pmatrix}\,.$$

to compute the eigenvalues of this matrix we search for the zeros of the characteristic polynomial, i.e.,

$$
\begin{aligned}
|A - \lambda \mathbb{1}| &= \begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} & \cdots & A_{1n} \\ & A_{22}-\lambda & A_{23} & \cdots & A_{2n} \\ & & A_{33}-\lambda & \cdots & A_{3n} \\ & & & \ddots & \vdots \\ & & & & A_{nn}-\lambda \end{vmatrix} \\
&= (A_{11}-\lambda) \begin{vmatrix} A_{22}-\lambda & A_{23} & \cdots & A_{2n} \\ & A_{33}-\lambda & \cdots & A_{3n} \\ & & \ddots & \vdots \\ & & & A_{nn}-\lambda \end{vmatrix} \\
&= (A_{11}-\lambda)(A_{22}-\lambda) \begin{vmatrix} A_{33}-\lambda & \cdots & A_{3n} \\ & \ddots & \vdots \\ & & A_{nn}-\lambda \end{vmatrix} \\
&= (A_{11}-\lambda)(A_{22}-\lambda)(A_{33}-\lambda)\cdots(A_{nn}-\lambda) \,.
\end{aligned}
$$

it follows that the eigenvalues coincide with the diagonal elements of the triangular matrix, i.e., the set of eigenvalues is

$$\{A_{11}, A_{22}, A_{33}, \ldots, A_{nn}\} \,.$$

eigenvalues, determinant, and trace

recall that the trace of a matrix is the sum of its diagonal elements, i.e.,

$$\operatorname{tr} A = A_{11} + A_{22} + \cdots + A_{nn} = \sum_{i=1}^{n} A_{ii} \,.$$

consider for consistency a (n-dimensional) vector space V over the field $\boxed{\mathbb{C}}$. let A be a linear map of V onto itself. then there exist $r \leq n$ eigenvalues of A,

$$\{\lambda_1, \lambda_2, \ldots, \lambda_r\} \,,$$

with algebraic multiplicities m_1, m_2, …, m_r, such that $m_1 + m_2 + \cdots + m_r = n$.

the eigenvalues $\{\lambda_1, \ldots, \lambda_r\}$ of a linear map A are intimately connected with its determinant and its trace: the determinant is the product of the eigenvalues, the trace is the sum of the eigenvalues. however, we must take care of the algebraic multiplicities; an eigenvalue λ_i with algebraic multiplicity m_i appears m_i times in the product or sum. therefore,

$$\det A = \underbrace{\lambda_1 \cdots \lambda_1}_{m_1 \text{ times}} \underbrace{\lambda_2 \cdots \lambda_2}_{m_2 \text{ times}} \cdots \underbrace{\lambda_r \cdots \lambda_r}_{m_r \text{ times}} = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_r^{m_r} = \prod_{i=1}^{r} \lambda_i^{m_i} \,,$$

$$\begin{aligned}\operatorname{tr} A &= \underbrace{\lambda_1 + \cdots + \lambda_1}_{m_1 \text{ times}} + \underbrace{\lambda_2 + \cdots + \lambda_2}_{m_2 \text{ times}} + \cdots + \underbrace{\lambda_r + \cdots + \lambda_r}_{m_r \text{ times}} \\ &= m_1\lambda_1 + m_2\lambda_2 + \cdots + m_r\lambda_r = \sum_{i=1}^{r} m_i \lambda_i \,.\end{aligned}$$

the proof is not difficult if one uses the characteristic polynomial and its decomposition into its roots, i.e.,

$$|A - \lambda \mathbb{1}| = (-1)^n \big(\lambda - \lambda_1\big)^{m_1} \big(\lambda - \lambda_2\big)^{m_2} \cdots \big(\lambda - \lambda_r\big)^{m_r} \,.$$

we restrict ourselves to the simple example of a (2×2) matrix, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \,.$$

let λ_1 and λ_2 denote the eigenvalues of A. (either $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$; in the latter case there exists only one eigenvalue whose multiplicity is 2). we obtain

$$\begin{aligned}|A - \lambda \mathbb{1}| = \begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix} &= (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} \\ &= \lambda^2 - \big(A_{11} + A_{22}\big)\lambda + A_{11}A_{22} - A_{12}A_{21} \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + \det A \\ (\lambda - \lambda_1)(\lambda - \lambda_2) &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \,.\end{aligned}$$

comparing the coefficients of the polynomials we are led to the result

$$\det A = \lambda_1 \lambda_2 \,, \qquad \operatorname{tr} A = \lambda_1 + \lambda_2 \,.$$

EXAMPLE

Consider the linear map on $\mathbb{C}^2$ represented by (2×2) matrix

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} .$$

We use $\det A$ and $\operatorname{tr} A$ to compute the eigenvalues of A.

$$\det A = 6\,, \qquad \operatorname{tr} A = 4\,.$$

Accordingly,

$$\lambda_1 \lambda_2 = \det A = 6\,, \qquad \lambda_1 + \lambda_2 = \operatorname{tr} A = 4\,,$$

which leads to a quadratic equation that can be solved to yield

$$\lambda_1 = 2 + \sqrt{2}\,, \lambda_2 = 2 - \sqrt{2}\,.$$

a simple corollary of the relation

$$\det A = \prod_{i=1}^{r} \lambda_i^{m_i}$$

is the statement that a linear map is singular (i.e., not invertible) if and only if zero is an eigenvalue of A.

3 Diagonalization

before we begin let us reiterate: a vector v of a vector space V can be represented by a column vector *once a basis of V has been chosen*. the column vector representation depends on the choice of basis. likewise, a linear map A of a vector space V onto itself can be represented as a matrix *once a basis of V has been chosen*. the matrix representation of A depends on the choice of basis.

Example

Consider the vector space $\mathbb{R}^2$ and the linear map A which describes a reflection at 45° (i.e., a reflection at the straight line with slope 45°). Under this reflection, the standard basis vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are mapped to

$$Ae_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2 \qquad Ae_2 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \,.$$

The matrix representation of A is obtained by using Ae_1 and Ae_2 as columns; hence

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,.$$

Now let us choose a different basis $\{b_1, b_2\}$ and represent the linear map A w.r.t. $\{b_1, b_2\}$. Choose

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,.$$

We obtain (from purely geometric considerations, i.e., by applying the reflection)

$$Ab_1 = b_1 \qquad Ab_2 = -b_2 \,.$$

Note that b_1 and b_2 are eigenvectors of A (associated with $\lambda_1 = 1$ and $\lambda_2 = -1$, respectively). Since we have chosen a (non-standard) basis, column vectors do not quite represent what we are used to. For instance, the vector

$$v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

now means $v = 2 \cdot b_1 + 0 \cdot b_2$, i.e., this vector v points along the 45° line. (It corresponds to $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ w.r.t. the old standard basis.)

To be continued...

EXAMPLE

... and now the conclusion. Likewise, the vector

$$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

now means $v = (-1)\cdot b_1 + 1\cdot b_2$, i.e., this vector v points in the direction of the negative x-axis. (It corresponds to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ w.r.t. the old standard basis.)

W.r.t. the new basis, the transformation $Ab_1 = b_1$, $Ab_2 = -b_2$, looks like

$$Ab_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b_1 \qquad Ab_2 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -b_2 \,.$$

The matrix representation of A is obtained by using Ae_1 and Ae_2 as columns; hence

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,.$$

We obtain a different matrix representation for the same linear map. This matrix representation is preferred to the original one since the matrix is *diagonal*.

we call a linear map A **diagonalizable**, if A possesses n linearly independent eigenvectors, i.e., a **basis of eigenvectors**.

when does this happen? suppose that

$$\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$$

are the eigenvalues of the linear map A. the associated eigenspaces (spaces of eigenvectors) are $E_1, E_2, \ldots, E_r$. the geometric multiplicity of the eigenvalue λ_i is $d_i = \dim E_i$. we know that there exist

$$d_1 + d_2 + \cdots + d_r \;\; (\leq n)$$

linearly independent eigenvectors. therefore, if and only if

$$d_1 + d_2 + \cdots + d_r = n \,,$$

then there exist n linearly independent eigenvectors. alternatively we can write

$$E_1 \oplus \cdots \oplus E_r = V \,.$$

an important case of diagonalizability is the case of a linear map A that possesses n different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (i.e., $r = n$). then, automatically, there exist n linearly independent eigenvectors. (this is simply because $d_i = \dim E_i \geq 1\ \forall i$; hence, if there exist n different eigenvalues, then $d_i = \dim E_i = 1\ \forall i$, and by the general considerations on linear independence, these eigenspaces/-vectors are linearly independent.)

let us suppose that the map A is diagonalizable and let us choose a basis of (i.e., n linearly independent) eigenvectors. we do this by successively choosing bases $\{v_{i;1}, \ldots, v_{i;d_i}\}$ in the eigenspaces E_i, i.e.,

$$V = \overbrace{\Big\langle \underbrace{v_{1;1}, \ldots, v_{1;d_1}}_{d_1 \text{ vectors}} \Big\rangle}^{E_1} \oplus \overbrace{\Big\langle \underbrace{v_{2;1}, \ldots, v_{2;d_2}}_{d_2 \text{ vectors}} \Big\rangle}^{E_2} \oplus \cdots \oplus \overbrace{\Big\langle \underbrace{v_{r;1}, \ldots, v_{r;d_r}}_{d_r \text{ vectors}} \Big\rangle}^{E_r} \,.$$

since $v_{i;j}$ is in E_i, it is an eigenvector associated with the eigenvalue λ_i, i.e.,

$$A v_{i;j} = \lambda_i v_{i;j} \,.$$

let us consider the matrix representation of the diagonalizable map A w.r.t. this basis of eigenvectors. let us denote the matrix we obtain by D (instead of A). we straightforwardly obtain

$$D = \operatorname{diag}\Big(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1 \text{ times}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{d_2 \text{ times}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r \text{ times}} \Big)$$

$$= \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_r & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_r \end{pmatrix} \,.$$

it therefore follows that a diagonalizable linear map can be represented by a **diagonal matrix**, whose entries are the eigenvalues. we call the diagonal matrix

$$D = \operatorname{diag}\big(\underbrace{\lambda_1,\dots,\lambda_1}_{d_1 \text{ times}},\underbrace{\lambda_2,\dots,\lambda_2}_{d_2 \text{ times}},\dots,\underbrace{\lambda_r,\dots,\lambda_r}_{d_r \text{ times}}\big)$$

the eigenvalue matrix of the linear map A.

conversely, if a map A can be represented by a diagonal matrix, then the eigenvalues are the entries of this matrix (so that the matrix is automatically the eigenvalue matrix) and the eigenvectors of A are represented by the column vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

hence, there exists a basis of eigenvectors and thus A is diagonalizable.

summing up, we see that a linear map A is **diagonalizable** if and only if it can be represented by a **diagonal matrix**.

Example

Consider the vector space $\mathbb{R}^3$ and the linear map A represented by the 'Sudoku matrix'

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} .$$

(The basis is tacitly assumed to be the standard basis.) The characteristic polynomial is

$$|A - \lambda \mathbb{1}| = -\lambda^3 + 15\lambda^2 + 18\lambda .$$

The eigenvalues are the zeros of the characteristic polynomial, i.e.,

$$\lambda_1 = 0 , \quad \lambda_2 = \tfrac{3}{2}\left(5 + \sqrt{33}\right) , \quad \lambda_3 = \tfrac{3}{2}\left(5 - \sqrt{33}\right) .$$

Since the map A has three different eigenvalues, it must have three linearly independent eigenvectors and thus a basis of eigenvectors. Therefore, the Sudoku map is diagonalizable and it can be represented by the diagonal eigenvalue matrix

$$D = \operatorname{diag}\left(0, \tfrac{3}{2}\,(5 + \sqrt{33}), \tfrac{3}{2}\,(5 - \sqrt{33})\right) .$$

EXAMPLE

Consider the vector space $\mathbb{C}^2$ and the linear map A represented by matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

(The basis is tacitly assumed to be the standard basis.) The eigenvalues can be read off directly, since this is a triangular matrix: There is only one eigenvalue, $\lambda = 1$. (Its algebraic multiplicity must be $m_\lambda = 2$.) Let us compute the space of eigenvectors E_λ. From

$$(A - \lambda \mathbb{1})\, v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we deduce that

$$E_\lambda = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle , \qquad d_\lambda = \dim E_\lambda = 1 .$$

In particular, there is only one eigenvector (and not two linearly independent ones). There does not exist a basis of eigenvectors; therefore, the map A is not diagonalizable.

EXAMPLE

In connection with the previous example we consider a rather trivial example: The identity map $\mathbb{1}$ has one eigenvalue, $\lambda = 1$ (with algebraic multiplicity $m_\lambda = 2$). Every vector is an eigenvector for $\mathbb{1}$, hence $E_\lambda = \mathbb{C}^2$ and $g_\lambda = \dim E_\lambda = 2$. The identity map is diagonalizable (and the standard matrix representation of $\mathbb{1}$ is already diagonal).

We see that it is not a problem if an eigenvalue appears multiple times (i.e., if its algebraic multiplicity is greater than 1). A problem occurs if the geometric multiplicity is strictly less than the algebraic multiplicity, $d_\lambda < m_\lambda$. In that case, $\sum_{i=1}^r m_i = n$ but $\sum_{i=1}^r d_i < n$, whence diagonalizability is ruled out.

EXAMPLE

Consider the vector space $\mathbb{R}^2$ and the linear map represented by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The characteristic polynomial is $\lambda^2 + 1 = 0$, hence there do not exist any eigenvalues.

If we consider the same map as a map on the vector space $\mathbb{C}^2$, then there exist two eigenvalues: $\lambda_1 = i$, $\lambda_2 = -i$. The map is not diagonalizable as a real map, but it is in fact diagonalizable regarded as a complex map. (Since there exist two different eigenvalues, there exist two linearly independent eigenvectors.)

in practice, a linear map is given in its matrix representation w.r.t. some (standard) basis,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} .$$

we know that, if and only if A is diagonalizable, then we can switch to a matrix representation in terms of a diagonal matrix (the eigenvalue matrix D). how do we switch in practice? we need a 'switch matrix' S.

the switch matrix is supposed to transform the standard basis to a basis of eigenvectors. on the basis of eigenvectors, the linear map then acts as a diagonal matrix (the eigenvalue matrix D). having applied the map in this simple form, we then switch back to the standard basis. hence,

$$D = S^{-1} A S .$$

the switch matrix contains the eigenvectors of A as columns, i.e.,

$$S = \begin{pmatrix} | & | & & | \\ v_{1;1} & v_{1;2} & \cdots & v_{r;d_r} \\ | & | & & | \end{pmatrix} .$$

to prove that $D = S^{-1}AS$ we show that $SDw = ASw$ for all $w \in V$. due to linearity, if we aim at proving a statement for all $w \in V$, it suffices to show this statement for all basis vectors. consider the standard basis vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

we obtain

$$Se_1 = v_{1;1} \quad \Rightarrow \quad ASe_1 = \lambda_1 v_{1;1} .$$

on the other hand,

$$De_1 = \lambda_1 e_1 \quad \Rightarrow \quad SDe_1 = \lambda_1 Se_1 = \lambda_1 v_{1;1} .$$

we conclude that $SDe_1 = ASe_1$; analogously, we obtain $SDe_i = ASe_i$ for all standard basis vectors e_i and thus $SDw = ASw$ for all $w \in V$. this completes the proof of the claim.

EXAMPLE

Consider the map A on $\mathbb{C}^3$ given by

$$A = \begin{pmatrix} 3+2i & -2-2i & -4 \\ 1+i & -i & -2 \\ 1+i & -1-i & -1 \end{pmatrix} .$$

The characteristic polynomial is

$$|A - \lambda \mathbb{1}| = -\lambda^3 + (2+i)\lambda^2 - (1+2i)\lambda + i \,;$$

it is not difficult to convince oneself that the factorization into roots is

$$|A - \lambda \mathbb{1}| = -(\lambda - i)(\lambda - 1)^2 .$$

Therefore, the eigenvalues are

$$\lambda_1 = i \quad (m_1 = 1, d_1 = 1)\,, \qquad \lambda_2 = 1 \quad (m_2 = 2)\,.$$

To see whether A is diagonalizable there must exist two linearly independent eigenvectors associated with the eigenvalue λ_2 (i.e., $d_2 = \dim E_2 = 2$ is required). A straightforward computation shows that

$$E_1 = \left\langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad E_2 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1-i \\ 0 \\ 1 \end{pmatrix} \right\rangle,$$

hence $d_2 = \dim E_2 = 2$ indeed; accordingly, there exist 3 linearly independent eigenvectors and A is diagonalizable.
The switch matrix S is

$$S = \begin{pmatrix} 2 & 1 & 1-i \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} ,$$

its inverse is

$$S^{-1} = \begin{pmatrix} -i & i & 1+i \\ i & 1-i & -1-i \\ i & -i & -i \end{pmatrix} .$$

It is straightforward to check that

$$S^{-1} A S = D = \operatorname{diag}(i, 1, 1) .$$