NOTES ON THE INITIAL VALUE PROBLEM FOR THE VACUUM EINSTEIN EQUATIONS

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Abstract. We here quickly study the initial value problem for the vacuum Einstein equations. Written in harmonic coordinates, the equations are hyperbolic, and we outline the main steps involved in proving existence and uniqueness of solutions. These notes are a summary of the treatment of [1, Chapters 1.1 & 1.2] (see, also, the early chapters of [2]).

1. Solving the initial value problem

We wish to construct a Lorentzian manifold $(M, \mathbf{g})$ where the metric $\mathbf{g}$ satisfies the vacuum Einstein equations $\mathbf{Ric}_{\mathbf{g}} = 0$. In particular, we want to rewrite this condition on $\mathbf{g}$ as a wave equation for the components of the metric with respect to a well-chosen coordinate system. In order to formulate and solve the initial value problem, we would like to find an appropriate coordinate system in which to study these equations.

1.1. **Harmonic coordinates.** Suppose we are given a coordinate system $\{x^a | a = 0, \dots, n\}$. The contravariant components of the metric are then $g^{ab} = \mathbf{g}(dx^a, dx^b)$. We wish to write the vacuum Einstein equations for the metric $\mathbf{g}$ as a wave equation for the functions g^{ab}. To this end, we denote by $\Box_{\mathbf{g}}$ the wave operator $g^{cd}\partial_c\partial_d$ acting on functions, i.e.

$$\Box_{\mathbf{g}} f \equiv g^{ab}\nabla_a\nabla_b f = g^{cd}\partial_c\partial_d f - g^{cd}\Gamma^a{}_{cd}\partial_a f.$$

We now consider $\Box_{\mathbf{g}}$ of the *functions* g^{ab}.

Proposition 1.1. *Let* $\lambda^a := \Box_{\mathbf{g}} x^a \equiv -g^{cd}\Gamma^a{}_{bc}$. *Then*

$$\Box_{\mathbf{g}} g^{ab} = \mathbf{g}(d\lambda^a, dx^b) + \mathbf{g}(dx^a, d\lambda^b) + 2\mathbf{g}(\nabla_c dx^a, \nabla^c dx^b) + 2\mathbf{Ric}(\nabla x^a, \nabla x^b). \tag{1.1}$$

Proof. We first note that, for any function f, we have

$$\Box_{\mathbf{g}}\nabla_a f = \nabla_c\nabla^c\nabla_a f = \nabla_c\nabla_a\nabla^c f = \nabla_a\nabla_c\nabla^c f + R_{ca}{}^c{}_d\nabla^d f \equiv \nabla_a\Box_{\mathbf{g}} f + \left(\mathbf{Ric}(\cdot, \nabla f)\right)_a.$$

Equivalently, in differential form notation,

$$\Box_{\mathbf{g}} df = d\left(\Box_{\mathbf{g}} f\right) + \mathbf{Ric}(\cdot, \nabla f).$$

In particular, taking f to be the coordinate functions x^a, we deduce that

$$\Box_{\mathbf{g}} dx^a = d\lambda^a + \mathbf{Ric}(\cdot, \nabla x^a).$$

We now simply calculate

$$\begin{aligned}\Box_{\mathbf{g}} g^{ab} &= \Box_{\mathbf{g}}\left(\mathbf{g}(dx^a, dx^b)\right)\\ &= \mathbf{g}(\Box_{\mathbf{g}} dx^a, dx^b) + \mathbf{g}(dx^a, \Box_{\mathbf{g}} dx^b) + 2\mathbf{g}(\nabla_c dx^a, \nabla^c dx^b)\\ &= \mathbf{g}(d\lambda^a, dx^b) + \mathbf{g}(dx^a, d\lambda^b) + 2\mathbf{Ric}(dx^a, dx^b) + 2\mathbf{g}(\nabla_c dx^a, \nabla^c dx^b).\end{aligned}$$ □

Treating the components of λ as a vector field, we have

$$\mathbf{g}(d\lambda^a, dx^b) = g^{bc}\partial_c\lambda^a = g^{bc}\left[\nabla_c\lambda^a - \Gamma^a{}_{cd}\lambda^d\right].$$

Denoting the contravariant components of the Ricci tensor by $R^{ab} \equiv \mathbf{Ric}(dx^a, dx^b)$, we therefore have the following.

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Proposition 1.2.

$$R^{ab} + \frac{1}{2}\left[\nabla^a\lambda^b + \nabla^b\lambda^a\right] = \tilde{E}^{ab}, \tag{1.2}$$

where

$$\tilde{E}^{ab} := \frac{1}{2}\Box_{\mathbf{g}} g^{ab} - \mathbf{g}(\nabla_c dx^a, \nabla^c dx^b) + \frac{1}{2}\left[g^{bc}\Gamma^a{}_{cd}\lambda^d + g^{ac}\Gamma^b{}_{cd}\lambda^d\right]$$

The point is to note that if we impose that $\tilde{E}^{ab} = 0$, this gives us the non-linear wave equation

$$\Box_{\mathbf{g}} g^{ab} = 2H^{ab}(g, \partial g, \lambda), \tag{1.3}$$

where

$$H^{ab}(g, \partial g, \lambda) := \mathbf{g}(\nabla_c dx^a, \nabla^c dx^b) - \frac{1}{2}\left[g^{bc}\Gamma^a{}_{cd}\lambda^d + g^{ac}\Gamma^b{}_{cd}\lambda^d\right],$$

which we may solve for the functions g^{ab}. Moreover, assuming that $\tilde{E}^{ab}$, we note that the metric $\mathbf{g}$ will satisfy the vacuum Einstein equations if we impose the condition that $\lambda^a = 0$, i.e. that the coordinates x^a are harmonic:

$$\Box_{\mathbf{g}} x^a = 0.$$

We quote the following general theorem from the theory of hyperbolic partial differential equations, which gives us existence of solutions to equation (1.3).

Theorem 1.3. *For any initial data*

$$g^{ab}(0, x^i) \in H^{s+1}, \qquad \partial_0 g^{ab}(0, x^i) \in H^s$$

with $s > \frac{n}{2}$, *on any open subset* $O \subseteq \{0\} \times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n$, *there exists a unique solution* $g^{ab}(t, x^i)$ *of* (1.3) *defined on a neighbourhood,* U, *of* O *in* $\mathbb{R} \times \mathbb{R}^n$.

As such, we have existence and uniqueness of solutions to (1.3). This will yield an existence result for the vacuum Einstein equations if we can show that we can consistently set $\lambda^a = 0$. The key observation is the following.

Proposition 1.4. *Assume that the metric* $\mathbf{g}$ *satisfies the equations* $\tilde{E}^{ab} = 0$ *in the coordinate system* x^a. *Then the functions* λ^a *satisfy the wave equation*

$$\Box_{\mathbf{g}} \lambda^a + R^a{}_b \lambda^b = 0. \tag{1.4}$$

Proof. Assume that we have solved the problem $\tilde{E}^{ab} = 0$ with a given λ. From (1.2), we deduce that the Ricci tensor of the metric $\mathbf{g}$ is given by

$$R^{ab} = -\frac{1}{2}\left[\nabla_a\lambda^b + \nabla_b\lambda^a\right].$$

This implies that $\mathbf{g}$ has scalar curvature

$$s_{\mathbf{g}} = -\nabla \cdot \lambda.$$

The Bianchi identities for $\mathbf{g}$ imply that

$$\begin{aligned}
0 &= \nabla_a\left(R^{ab} - \frac{s_{\mathbf{g}}}{2} g^{ab}\right) \\
&= -\frac{1}{2}\nabla_a\left(\nabla^a\lambda^b + \nabla^b\lambda^a - (\nabla\cdot\lambda)\, g^{ab}\right) \\
&= -\frac{1}{2}\left(\Box\lambda^b + \nabla_a\nabla^b\lambda^a - \nabla^b\nabla_a\lambda^a\right) \\
&= -\frac{1}{2}\left(\Box\lambda^b + R_a{}^{ba}{}_c\lambda^c\right), \\
&= -\frac{1}{2}\left(\Box\lambda^b + R^b{}_c\lambda^c\right),
\end{aligned}$$

which gives the required result. □

As such, if the g^{ab} evolve according to $\tilde{E}^{ab} = 0$, then the λ^a also evolve according to the wave equation (1.4). It follows from the standard theory of hyperbolic PDE's that if we set initial data on a spacelike hypersurface O with $\lambda|_O = 0$ and $\mathbf{e}_0\lambda|_O = 0$ (where $\mathbf{e}_0$ is the timelike normal to O) then the unique solution of (1.4) on the domain of dependence of O, $\mathscr{D}(O)$, is $\lambda^a = 0$. Therefore, the solution of $\tilde{E}^{ab} = 0$ that we construct using Theorem 1.3 gives a metric written in harmonic coordinates that satisfies the vacuum Einstein equations.

1.2. **Constraints on the initial data.** We have previously seen that, when we do a decomposition of the vacuum Einstein equations into time and spatial parts, then the data on a space-like hypersurface must satisfy certain constraints. We now quickly discuss where these constraints arise in the current formulation.

We may choose the harmonic coordinates x^a such that $x^0 = t$, where the initial hypersurface where we set our data is the set $t = 0$. The constraint that $\lambda^a = 0$, i.e. $\Box_{\mathbf{g}} x^a = 0$, implies that we have

$$0 = \Box_{\mathbf{g}} x^a = \frac{1}{\sqrt{-\det g}}\partial_b\left(\sqrt{-\det g}g^{bc}\partial_c x^a\right) = \frac{1}{\sqrt{-\det g}}\partial_b\left(\sqrt{-\det g}g^{ba}\right).$$

Therefore we deduce that

$$\partial_t\left(\sqrt{-\det g}g^{ta}\right) = \partial_i\left(\sqrt{-\det g}g^{ia}\right). \tag{1.5}$$

This identity must hold on the initial surface $t = 0$. As such, there are non-trivial relationships between the initial data for g^{ab} and $\partial_t g^{ab}$ at $t = 0$, and these data cannot be chosen arbitrarily. More specifically, (1.5) effectively determines the values of $\partial_t g^{ta}$ at $t = 0$ once we have prescribed the components g^{ia} at $t = 0$. In particular, we may choose the coordinates so that the metric satisfies the conditions

$$g^{tt} = -1, \qquad g^{ti} = 0 \tag{1.6}$$

at $t = 0$. In this case, prescribing the g_{ij} and $\partial_t g_{ij}$ at $t = 0$ determines the time derivatives of g^{ta} at $t = 0$. It follows that the initial data are completely determined by the spatial metric

$$\mathbf{h} := g_{ij}dx^i \otimes dx^j$$

and its time derivative, i.e. the first and second fundamental form of the hypersurface $t = 0$.

If we assume that the metric components evolve according to (1.3), i.e. $\tilde{E}^{ab} = 0$, then the components of the Einstein tensor of $\mathbf{g}$ are given by

$$G^{ab} = -\frac{1}{2}\left[\nabla^a\lambda^b + \nabla^b\lambda^a - (\nabla\cdot\lambda)\, g^{ab}\right].$$

If we, for example, impose the metric conditions (1.6) and that $\lambda^a \equiv 0$ when $t = 0$ (and hence that the spatial derivatives of the λ^a are zero at $t = 0$), then we have, for example,

$$G^{tt} = -\frac{1}{2}\left[2\nabla^t\lambda^t + \nabla\cdot\lambda\right] = -\frac{1}{2}\left[-2\partial_t\lambda^t + \partial_t\lambda^t\right] = \frac{1}{2}\partial_t\lambda^t.$$

Therefore, the condition that $\partial_t\lambda^t = 0$ at $t = 0$ is equivalent to the scalar constraint $G^{tt} = 0$. Similarly, at $t = 0$, we have

$$G^{ti} = -\frac{1}{2}\left[\nabla^t\lambda^i + \nabla^i\lambda^t - (\nabla\cdot\lambda)\, g^{ti}\right] = \frac{1}{2}\partial_t\lambda^i.$$

Therefore, the constraint G^{ti} is equivalent to the condition that $\partial_t\lambda^i = 0$ at $t = 0$.

References

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