

Probing the Big-Bang With Quantum Fields

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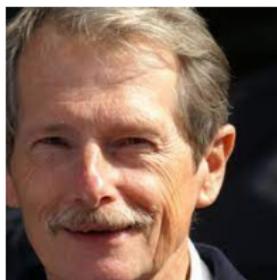
Joint work with **De Lorenzo, Del Rio, Schneider**.

First paper: ATMP (at Press); Second: (in preparation);

Uses earlier work with Henderson & Sloan (PRD & CQG).

Peter Aichelburg Fest, 4th November 2021

To my Colleague and Dear Friend,
Peter Aichelburg:
Thank you for your wry sense of humor,
Your wisdom,
Your wonderfully balanced views,
and thank you and Hanni for your
gracious hospitality spanning decades!



HAPPY 80TH PETER!!
May this tribute via
distributional geometries
delight you.

Preamble

- Space-like singularities are taken to be the absolute beginning or end of space-time. Geodesics of test particles end there. Tidal forces between them become infinite. But what if one uses **quantum** probes? It has been long argued that they may be tamer for physically more realistic probes. Examples:

Horowitz and Marolf (1995): In certain **static** space-times with **time-like singularities**: Dynamics of test quantum particles well-defined in some cases.)

Stalker & Tahvildar-Zadeh (2004): In **Reissner-Nordstrom** space-times dynamics of classical fields well-defined across the **time-like singularities**.

Hofmann and Schneider (2015): The **Schwarzschild space-like singularity** probed with test quantum fields. Found to be tame. But the **arguments are formal**; infinite number of degrees of freedom did not receive due care.

- Goal: Revisit the issue for the physically most important **dynamical** singularities with precision required to handle the **infinite number of DOF** of QFT carefully. Due to time limitation, this talk focuses on the Big Bang/Big Crunch singularities in the FLRW models. But the approach is general.

- Main Question: Do test quantum fields $\hat{\phi}(x)$ and observables constructed from them such as $\langle \hat{\phi}(x) \hat{\phi}(x') \rangle$, $\langle \hat{\phi}^2(x) \rangle_{\text{ren}}$, $\langle \hat{T}_{ab}(x) \rangle_{\text{ren}}$ remain regular **in the sense of QFT** across the big bang? Surprisingly, the answer is **in the affirmative!**

Nature of Quantum Fields: Minkowski Space

- The Quantum Field in Minkowski space (\hat{M}, \hat{g}_{ab}) is an **operator-valued (tempered) distribution (OVD)** :

$$\hat{\phi}^\circ(x) = \int \frac{d^3k}{(2\pi)^3} [\hat{A}(\vec{k}) \frac{e^{-i\omega\eta}}{\sqrt{2\omega}} + \hat{A}^\dagger(-\vec{k}) \frac{e^{i\omega\eta}}{\sqrt{2\omega}}] e^{i\vec{k}\cdot\vec{x}} \quad \text{with } x \equiv (\vec{x}, \eta)$$

satisfying $(\square - m^2)\hat{\phi}^\circ(x) = 0$. That is,

$\hat{\phi}^\circ(f) = \int_{\hat{M}} d^4\hat{V} \hat{\phi}^\circ(x)f(x)$ is a SA operator on the Fock space $\hat{\mathcal{F}}$ satisfying $\int_{\hat{M}} d^4\hat{V} \hat{\phi}^\circ(x)(\square - m^2)f(x) = 0$, for all test functions $f(x) \in \mathcal{S}$, the Schwartz space.

- The distributional character is not a mere technicality but is conceptually important. For example in

$$\begin{aligned} [\hat{\phi}^\circ(x), \hat{\phi}^\circ(x')] &= i\hbar (G_{\text{ad}} - G_{\text{ret}})(x, x') \hat{I} \quad \text{and} \\ \langle \hat{\phi}^\circ(x) \hat{\phi}^\circ(x') \rangle_\circ &= \frac{\hbar}{4\pi^2} \frac{1}{|\vec{x}-\vec{x}'|^2 - ((\eta-\eta')-i\epsilon)^2} \end{aligned}$$

the right sides are genuine distributions; not functions. Meaning of $i\epsilon$: first integrate $\langle \hat{\phi}^\circ(x) \hat{\phi}^\circ(x') \rangle_\circ$ with $i\epsilon$ with the test functions and then take the limit. More importantly, products $\hat{\phi}^2(x)$ have to be regularized precisely because $\hat{\phi}^\circ(x)$ is an OVD. **The textbook terminology of 'field operators' and '2-point functions' (and Dirac ' δ function')** can be very misleading if taken literally.

FLRW space-times

- So the question is: Do quantum fields $\hat{\phi}(x)$ continue to be well-defined across the big-bang as OVDs? For example, in 3-d, $1/r$ is singular as a function but a regular as a tempered distribution (satisfying the well-known equality $\vec{\nabla}^2(1/r) = 4\pi\delta^3(\vec{r})$).

- Recall that the Friedmann, Lemaître, Robertson, Walker (FLRW) space-time (M, g_{ab}) is conformally flat:

$$g_{ab}dx^a dx^b = a^2(\eta) \dot{g}_{ab} dx^a dx^b \equiv a^2(\eta) (-d\eta^2 + d\vec{x}^2) \quad \text{with } a(\eta) = a_\beta \eta^\beta; \beta \geq 0.$$

$\eta > 0$ on M , and the big bang corresponds to $\eta = 0$. We can extend $a^2(\eta)$ and hence g_{ab} to the full Minkowski manifold $\overset{\circ}{M}$ with $\eta \in (-\infty, \infty)$ as a continuous tensor field (albeit degenerate at $\eta = 0$). Systematic Rationale:

- There is a generalization of the ADM framework for the initial value problem of full GR (based on 'connection dynamics' in which the metric arises as an emergent, secondary field), which is equivalent to the ADM framework when the 3-metric is non-degenerate, but which does not break down if it becomes degenerate. (AA, Henderson & Sloan). In FLRW (as well as Bianchi models and the Schwarzschild solution) it enables one to evolve across the singularity unambiguously (Kosłowski, Mercati & Sloan; Mercati, AA & Valdes). (Recall: 'Hubble-normalized variables' used in cosmology.) In FLRW models, the extension yields just the simple prescription given above. As a tensor field, g_{ab} is C^0 at $\eta = 0$ & smooth if $\eta \neq 0$.

QFT in FLRW Space-times

- For definiteness, consider the massless scalar field: $\square \hat{\phi} = 0$.

On M , $\phi^\circ = a(\eta)\phi(x)$ satisfies a simple equation with respect to the Minkowski metric \hat{g}_{ab} in presence of a 'universal' time dependent potential:

$$(\hat{\square} - V(\eta))\hat{\phi}^\circ(x) = 0 \quad \text{with} \quad V(\eta) = \beta(\beta - 1)/\eta^2.$$

Rigorous Result: Because of the form of the potential, one can introduce a **canonical** \pm frequency decomposition (i.e. a **canonical** Kähler structure) on the space of classical solutions and write the general solution as

$$\phi^\circ(x) = \int \frac{d^3k}{(2\pi)^3} [z(\vec{k})\hat{e}(k, \eta) + z^*(-\vec{k})\hat{e}^*(k, \eta)] e^{i\vec{k}\cdot\vec{x}}$$

where $\hat{e}(k, \eta)$ are the positive-frequency modes and $z(\vec{k})$ are regular coefficients (in S). Then the putative OVD on FLRW space-time is given by:

$$\hat{\phi}(x) = \frac{1}{a(\eta)}\hat{\phi}^\circ(x) = \frac{1}{a(\eta)} \int \frac{d^3k}{(2\pi)^3} [\hat{A}(\vec{k})\hat{e}(k, \eta) + \hat{A}^\dagger(-\vec{k})\hat{e}^*(k, \eta)] e^{i\vec{k}\cdot\vec{x}}.$$

- The mode functions $\hat{e}(k, \eta)$ are explicitly known. Generically they diverge at $\eta = 0$. For example, for dust ($\beta = 2$), they are $(e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$. As **functions**, they diverge at the big bang. And there is another $1/a(\eta)$ overall factor in $\hat{\phi}(x)$. The question is: Is $\hat{\phi}(x)$ nonetheless well-defined as an OVD across the big bang on full \hat{M} ?

Radiation-filled Universe

• Is $\hat{\phi}(f) = \int_{\dot{M}} d^4V \hat{\phi}(x) f(x)$ satisfying $\int_{\dot{M}} d^4V \hat{\phi}(x) (\square f) = 0 \quad \forall f \in \mathcal{S}$ well-defined on the extended space-time, i.e. for all $\eta \in (-\infty, \infty)$?

• Radiation-filled universe: $a(\eta) = a_1 \eta$ (i.e. $\beta = 1$) $\Rightarrow V(\eta) = 0$, whence, $\hat{\phi}^\circ(x) = a(\eta) \hat{\phi}(x)$ now satisfies $\square \hat{\phi}^\circ = 0$ in Minkowski space! Hence mode functions $\hat{e}(k, \eta)$ same as in Minkowski space; trivially regular for $\eta \in (-\infty, \infty)$. But the physical field on FLRW space-time is $\hat{\phi}(x) = a^{-1}(\eta) \hat{\phi}^\circ(x)$ and $a(\eta) = 0$ at $\eta = 0$!

How could it then be regular on Minkowski Fock space?

Answer: The physical volume element is $d^4V = a^4 d^4x$. Hence:

$$\hat{\phi}(f) = \int_{\dot{M}} d^4V \hat{\phi}(x) f(x) = \int_{\dot{M}} d^4x \hat{\phi}^\circ(x) (a^3(\eta) f(x))$$

and $a^3(\eta) f(x) \in \mathcal{S}$ if $f \in \mathcal{S}$. Hence $\hat{\phi}(f)$ is in fact a **well-defined operator** on the Minkowski Fock space $\Rightarrow \hat{\phi}(x)$ well defined OVD for all $\eta \in (-\infty, \infty)$!

Next, the expectation value of the product of fields

$$\langle \hat{\phi}(x) \hat{\phi}(x') \rangle = \frac{1}{a_1^2 \eta \eta'} \langle \hat{\phi}^\circ(x) \hat{\phi}^\circ(x') \rangle_\circ = \frac{1}{a_1^2 \eta \eta'} \frac{\hbar}{4\pi^2} \frac{1}{r^2 - (t - i\epsilon)^2}$$

(where $r = |\vec{x} - \vec{x}'|$ and $t = \eta - \eta'$) **is also a well-defined bi-distribution** because $d^4V = a_1^4 \eta^4 d^4x$ and d^4x is well defined on all of \dot{M} . Furthermore $\langle \hat{\phi}^2(x) \rangle_{\text{ren}}$ and $\langle \hat{T}_{ab}(x) \rangle_{\text{ren}}$ are both well-defined distributions on all of the extended manifold \dot{M} . $\langle \hat{T}_{ab}(x) \rangle_{\text{ren}}$ is non-trivial. So the theory is not trivial.

Dust-filled Universe

- More interesting case: Dust-filled universe where $a(\eta) = a_2\eta^2$. Hence $(1/a(\eta))$ diverges faster at $\eta = 0$ and **mode functions** $(e^{ik\eta}/\sqrt{2k})(1 - i/k\eta)$ **also diverge at the big bang** (unlike in the radiation-filled case).

- The 1-particle Hilbert space is built out of solutions:

$$\phi(x) = \frac{1}{a(\eta)} \int \frac{d^3k}{(2\pi)^3} [z(\vec{k})\hat{e}(k, \eta) + z^*(-\vec{k})\hat{e}^*(k, \eta)] e^{i\vec{k}\cdot\vec{x}} \text{ with } z(\vec{k}) \in \tilde{\mathcal{S}}$$

and they all diverge at $\eta = 0$. So how can there be a well-defined Fock space? There is, because the 1-particle norm $\|\phi(x)\|$ is perfectly finite (and non-zero) at $\eta = 0$ because the divergence in the value of $\phi(x)$ is **precisely** compensated by the vanishing of the 3-volume element there!

$$\|\phi(x)\|^2 = \frac{1}{\hbar} \int \frac{d^3K}{(2\pi)^3} |z^*(\vec{k})|^2.$$

This is analogous to the QM fact that while $\Psi(\vec{x}) := (1/r)e^{-\alpha r}$ is divergent as a function, it represents a well-defined state in $\mathcal{H} := L^2(\mathbb{R}^3)$, because the volume element d^3x goes as r^2 .

- Since $d^4V = a_2^4\eta^8 d^4x$, $\hat{\phi}(f)$ is a well-defined OVD. (However, there is an infrared subtlety (Ford and Parker (1977)). Already for $\eta > 0$, the action of $\hat{\phi}(x)$ is well-defined on a co-dimension 1 subspace \mathcal{S}_1 of \mathcal{S} and there is a 1-parameter freedom in extending its action on full \mathcal{S} , representing an infrared cutoff ℓ . But, this cutoff has nothing to do with the big bang. Once $\hat{\phi}(x)$ defined for $\eta > 0$, it continues to be well-defined for $\eta \leq 0$).

Dust filled Universe: 'BKL Behavior'

- The expectation value of the product of fields is given by

$$\langle \hat{\phi}(x) \hat{\phi}(x') \rangle = \frac{\hbar}{4\pi^2} \frac{1}{a_2^2 \eta^2 \eta'^2} \left[\frac{1}{(r^2 - (t - i\epsilon)^2)} + \frac{1}{\eta \eta'} [2(1 - \gamma) + \ln \frac{r^2 - (t - i\epsilon)^2}{\ell^2}] \right].$$

It is a well-defined bi-distribution, i.e., $\int_{\overset{\circ}{M}} d^4V d^4V' \langle \hat{\phi}(x) \hat{\phi}(x') \rangle f_1(x) f_2(x)$ is well-defined because $d^4V = a_2^4 \eta^8 d^4x$, and d^4x is well-defined on all of $\overset{\circ}{M}$.

For space-like and time-like separated points, one interprets $\langle \hat{\phi}(x) \hat{\phi}(x') \rangle$ as a **correlation function**. In Minkowski space, correlations decay as $1/\text{Dist}^2$ for both space-like and time-like separations.

- Now, there is an interesting space vs time asymmetry as one approaches the singularity: Consider points that are **space-like or time-like** separated by a **fixed proper (geodesic) distance D** . As one approaches the big bang, but space-like correlations dominate over but time-like ones:

$\lim_{\eta_0 \rightarrow 0} \frac{\langle \hat{\phi}(\vec{x}, \eta_0) \hat{\phi}(\vec{x}', \eta_0) \rangle}{\langle \hat{\phi}(\vec{x}_0, \eta_0) \hat{\phi}(\vec{x}_0, \eta) \rangle} = \infty$ (as $2D/a_2\eta^3$). Strong correlations \sim smaller variations \Rightarrow smaller derivatives.

Therefore, **"time derivatives dominate over space-derivatives"** as in the well-known BKL behavior of GR. But one has to keep in mind that conceptually these are quite different statements: this behavior refers to test quantum fields on a given FLRW background while the BKL behavior refers to the gravitational field itself.

Summary and Generalizations

- Summary: The big bang and big crunch singularities are remarkably tamer when probed with observables associated with **quantum** fields. The classical fields $\phi(x)$ that define 1-particle states do diverge at the big bang singularity. But their norm in the 1-particle Hilbert space is **finite** because the shrinking of the volume element exactly compensates for this divergence. Similar to the fact that the wave function $\Psi(\vec{r}) = (1/r)e^{-r/r_0}$ diverges at the origin but is a well-defined element of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ in QM. Similarly, the mode functions that enter the expansion of $\hat{\phi}(x)$ diverge but **it is a well-defined OVD**: smeared operators $\hat{\phi}(f)$ are well defined. $\langle \hat{\phi}(x)\hat{\phi}(x') \rangle$ –**and even** $\langle \hat{\phi}^2(x) \rangle_{\text{ren}}$ **and** $\langle \hat{T}_{ab}(x) \rangle_{\text{ren}}$ – are well-defined tempered distributions, just as they are in Minkowski space.
- Generalization: The main result on tame behavior of linear, test quantum fields extend to other FLRW models; I used Radiation and dust filled cases because the mode functions are sufficiently simple to display explicit results.
- Higher spins: Since FLRW space-times are conformally flat, quantum (as well as classical) Maxwell fields are trivially regular across the big bang and big crunch. Results on the massless scalar field imply that tameness will persist also for spin 2 (i.e. linearized gravitational) fields.

Broader Perspective

- Investigations of whether the most important singularities of GR appear tame because of the ‘fuzziness’ associated with quantum probes have a long history. Detailed analysis showed that the answer is in the affirmative if the big bang (and also the Schwarzschild) singularity is probed with quantum fields that are inherently distributional. These results provide hints for a full quantum gravity theory: to obtain a self-consistent theory that allows matter and geometry to interact quantum mechanically, geometry should also have a **distributional character** at the micro level. This feature arises in diverse approaches to quantum gravity where, in the Planck regime, excitations of quantum geometry have support in 2 (space-time) dimensions. A concrete example is provided by the distributional nature of quantum geometry in loop quantum gravity and spinfoams. One can now hope to bridge QFT in CST with full quantum gravity.

Peter: You have shed much light on the physics of distributional space-time geometries! Hope these hints that geometry is distributional at a fundamental micro level delight you!

Many Happy Returns of the Day!

