Exercises in Relativity and Cosmology II summer term 2016

Problem 50

Show $\mathcal{L}_u v = [u, v]$ and $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$.

Problem 51

Show: If there exists a nontrivial Killing vector field ξ, then there exist coordinates such that g_{ik} does not depend on one of them.

Problem 52

The Gauss theorem for an antisymmetric tensor field J^{ik} defined on a hypersurface Σ with boundary reads

$$\int_\Sigma \nabla_i J^{ik} d\sigma_k = -\frac{1}{2}\int_{\partial\Sigma} J^{ik} d\sigma_{ik}$$

$$(d\sigma_{ik} = \frac{1}{2}\epsilon_{iklm} dx^l \wedge dx^m).$$

Use this theorem to show that a spacetime with Killing vector ξ possesses the conserved quantity

$$Q_\xi := -\frac{1}{\kappa}\int_{S_\infty} \nabla^i \xi^k d\sigma_{ik},$$

where S_∞ is the 'boundary at infinity' of the spacelike hypersurface Σ. Use the Einstein field equations to find a relation between this conserved quantity and $\int_\Sigma T^{ik}\xi_i d\sigma_k$. What is Q_ξ in the case of the timelike Killing vector of the Schwarzschild metric

$$ds^2 = (1 - \frac{2\mathcal{M}}{r})dt^2 - \frac{1}{1-\frac{2\mathcal{M}}{r}}dr^2 - r^2 d\Omega^2 \ ?$$

Problem 53

Prove: The Killing vector ξ is hypersurface orthogonal $\Leftrightarrow \nabla_{[j}(\frac{\xi_{k]}}{\xi^2}) = 0$.

Problem 54

The GPS satellites are in a circular orbit 20200 km above the surface of the earth. A GPS receiver at rest on the surface of the earth compares the received satellite clock signals with its own clock. What is the ratio of corresponding time intervals? Estimate from this the error in position determination that would arise in the course of a day from neglecting relativistic corrections.

Problem 55

Compute the mean curvature of a surface embedded in $\mathbf{E}^3$. Use the result of Problem 43 to verify the scalar constraint equation. Show that in two dimensions this equation reduces to $K_G = det(K_{ij})$, where K_G is the Gaussian curvature.

Problem 56

Prove $K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij}$ for a hypersurface-orthogonal continuation of n.

Problem 57

Prove the formula that expresses the covariant derivative D on a hypersurface in terms of the 4-dimensional covariant derivative ∇. Hint: Use the fundamental theorem.

Problem 58

Derive $\nabla_k T^{ik} = 0$ from the invariance of the matter action under (infinitesimal) diffeomorphisms.

Problem 59

Generalize the special-relativistic Maxwell action

$$S[A] = -\frac{1}{16\pi} \int d^4x F_{ij} F^{ij} + \int d^4x j^i A_i$$

to a curved spacetime and derive from this generalization the covariant Maxwell equations and the metric energy-momentum tensor.