Exercises in Relativity and Cosmology II summer term 2015

Problem 25
The Maxwell equations in the Lorenz gauge follow from the Lagrangian $\mathcal{L} = \frac{1}{8\pi} A_{i,k} A^{i,k} - j^i A_i$. Show that the canonical energy-momentum tensor of the electromagnetic field derived from this Lagrangian coincides with the Maxwell tensor in the case of a plane wave in the radiation gauge.

Problem 26
Construct a Lagrangian for the linearized gravitational field ψ_{ik} in the harmonic gauge analogous to the Lagrangian of Problem 25. (with interaction term $-\psi_{ik} T^{ik}$) and compute from it the energy-momentum tensor of a plane wave in the TT gauge. Compare the result with that derived in the lecture.

Problem 27
Estimate the energy-current density of a gravitational wave with amplitude $h \sim 10^{-20}$ and frequency ~ 100Hz and compare it with that of moonlight.

Problem 28
What is the gravitational radiation power of a satellite of mass m in circular orbit around a central mass M at a distance r? What is the frequency of the radiation? Provide the numbers for the example of the earth orbiting the sun.

Problem 29
Derive an upper limit for the radiation power of any bound system (use $V_0 \leq c^2$).

Problem 30
Let M_1 be the real line $\mathbb{R}$ with the standard differentiability structure and M_2 the differentiable manifold defined by $\mathbb{R}$ with the atlas $\{x \mapsto x^3\}$. Show:
a) The differentiability structures of M_1 and M_2 are not compatible.
b) The identity on $\mathbb{R}$ is not a diffeomorphism from M_1 to M_2.
c) Give an example of a diffeomorphism from M_1 to M_2.

Problem 31
Show that the abstract definition of a tangent vector v implies that $v(c) = 0$ for a constant function c.

Problem 32
Show that the tangent vectors $\partial_i|_P$ form a basis for T_P. Use and prove the following lemma: An arbitrary function $\phi \in C^\infty(\mathbb{R}^n)$ can be written as $\phi(x) = \phi(0) + g_i(x)x^i$ with $g_i(0) = \partial_i \phi|_0$.

Problem 33
Show abstractly that the Lie bracket of two vector fields is a vector field and compute the components of $[u, v]$.