Exercises in Relativity and Cosmology II summer term 2014

Problem 13
Prove the Bianchi identity.

Problem 14
Show that in 4 dimensions $T_{ik} \to T_{ik} - \frac{1}{2} g_{ik} T^l{}_l$ is an involution in the space of symmetric tensors. How does this generalize to n dimensions?

Problem 15
In a Newtonian gravitational field test particles are at rest at $t = 0$ with respect to a freely falling observer. They populate the surface of a small domain $\mathcal{V}$ of volume δV that contains the observer and is free of sources. Show that the volume enclosed by the particles stays initially approximately constant: $\delta V(t) = \delta V(0)$ for small t. Sketch the time evolution of a spherical domain $\mathcal{V}$ in a central field (the center of the field is outside $\mathcal{V}$).

Problem 16
Prove this generalization of the statement of Problem 15:
In a general Newtonian gravitational field

$$\frac{d^2 \delta V}{dt^2}|_{t=0} = -4\pi G \delta M,$$

where δM is the total mass contained in the domain $\mathcal{V}$ enclosed by the particles.

Problem 17
Prove the general relativistic generalization of the statement of Problem 16:

$$\frac{D^2 \delta V}{d\tau^2}|_{\tau=0} = R_{ij} u^i u^j \delta V,$$

where τ and u^i are the proper time and 4-velocity, respectively, of the freely falling observer. *Hint:* Note that δV is measured in the rest system of the observer.

Problem 18
By comparing the result of 17. with that of 16. conclude that $-(4\pi G)^{-1} R_{00}$ has the meaning of the *density of active gravitational mass* and show that in a local Lorentz system

$$R_{00} = -\frac{\kappa}{2}(\epsilon + p_1 + p_2 + p_3).$$

What is the meaning of the p_α?

Problem 19
Show that $R_{ik} = -\kappa T_{ik}$ implies the constancy of $T^l{}_l$.

Problem 20
Show that the harmonic gauge condition can always be fulfilled locally. What is the remaining gauge freedom?

Problem 21

Deduce the harmonic gauge condition from the harmonic coordinate condition.

Problem 22

Show that the linearized Riemann tensor is gauge invariant.

Problem 23

Prove the *Laue theorem*:

$$\int T_{\alpha\beta} d^3x = \frac{1}{2}\frac{d^2}{dt^2}\int T_{00} x^\alpha x^\beta d^3x$$

for an isolated system in Minkowski space.

Problem 24

Show with the help of the Laue theorem that every isolated stationary energy-momentum distribution generates the asymptotic field $h_{00} = GM_{in}/r$. Why does this not contradict the formal result $h_{00} = 2GM_{in}/r$ for a pure electromagnetic field?