1 Some timelike geodesics in Schwarzschild. Let $V^2 = 1 - 2m/r$, and consider the Schwarzschild metric:

$$g = -(1 - \tfrac{2m}{r})dt^2 + \tfrac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2 \,, \tag{1}$$

$$t \in \mathbb{R} \,,\; r \neq 2m, 0 \,, \tag{2}$$

Recall, from a previous problem sheet, that for geodesics in the Schwarzschild metric lying in the equatorial plane $\theta = \pi/2$ we have the following constants of motion (you are strongly encouraged to check that you can reproduce the result, but this will not be done again in class):

$$\frac{d}{ds}\left(V^2 \frac{dt}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{dt}{ds} = \frac{E}{1 - \frac{2m}{r}} \,. \tag{3}$$

$$\frac{d}{ds}\left(r^2 \frac{d\varphi}{ds}\right) = 0 \quad \Longrightarrow \quad \frac{d\varphi}{ds} = \frac{J}{r^2} \,. \tag{4}$$

$$\underbrace{V^2 \left(\frac{dt}{ds}\right)^2}_{E^2 V^{-2}} - V^{-2}\left(\frac{dr}{ds}\right)^2 - \underbrace{r^2 \left(\frac{d\varphi}{ds}\right)^2}_{J^2/r^2} = \lambda \in \{0, \pm 1\} \,. \tag{5}$$

Consider a timelike geodesic in the Schwarzschild metric parameterized by proper time, thus $\lambda = 1$, and lying in the equatorial plane $\theta = \pi/2$.

Q1. Verify that

$$\frac{E^2 - \dot{r}^2}{1 - \frac{2m}{r}} - \frac{J^2}{r^2} = 1 \,.$$

Q2. Deduce that if $E = 1$ and $J = 4m$ then

$$\frac{\sqrt{r} - 2\sqrt{m}}{\sqrt{r} + 2\sqrt{m}} = A e^{\epsilon\varphi/\sqrt{2}} \,,$$

where $\epsilon = \pm 1$ and A is a constant. Describe the orbit that starts at $\varphi = 0$ in each of the cases (i) $A = 0$, (ii) $A = 1$, $\epsilon = -1$, (iii) $r(0) = 3m$, $\epsilon = -1$.

Q3: Consider the case $E = 1$, $J = 0$ and deduce a relation for $r = r(s)$.

2 Null Geodesics in Schwarzschild Let γ be an affinely parameterized null geodesic, thus $\lambda = 0$, in the Schwarzschild metric lying in the equatorial plane $\theta = \pi/2$. Assume that $J \neq 0$, hence we can make a change of parameter $\varphi \mapsto s(\varphi)$ using the implicit equation

$$\frac{d\varphi}{ds} = \frac{J}{r^2} \quad \Longrightarrow \quad \frac{ds}{d\varphi} = \frac{r^2}{J} \,.$$

Set

$$u(\varphi) = \frac{m}{r(s(\varphi))}\ , \quad p(\varphi) = \frac{du(\varphi)}{d\varphi}\ .$$

Q1. Show that along γ, for $u \neq \frac{1}{2}$, we have

$$p^2 = 2u^3 - u^2 + \alpha^2\ . \tag{6}$$

Q2. For the case $\alpha = 0$ show that we obtain an autonomous two-dimensional system

$$\frac{du}{d\varphi} = p\ , \quad \frac{dp}{d\varphi} = 3u^2 - u\ .$$

Recall that critical points of a dynamical system $d\vec{x}/d\varphi = \vec{Y}$ (here $\vec{x} = (u, p)$) are defined as points where $\vec{Y}$ vanishes. Find the critical points. Can you sketch the trajectories in the (u, p) phase-plane?

Q3. For the case $\alpha = 0$ and $u > 1/2$ show that the geodesic has an equation of the form

$$r = 2m\cos^2\left(\frac{\varphi - \varphi_0}{2}\right)\ , \tag{7}$$

with $t = t(\varphi)$ that you should determine.