1 a) Show that for any connection we have

$$\nabla_\alpha \delta^\beta_\gamma = 0.$$

Hence, or otherwise, show that for the Levi-Civita connection it holds that

$$\nabla_\alpha g^{\beta\gamma} = 0.$$

b) Show that

$$\Gamma^\alpha_{\mu\alpha} = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu\left(\sqrt{|\det g_{\alpha\beta}|}\right) .$$

Find likewise a simple expression for $g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$, and deduce that

$$\Box_g f := g^{\mu\nu}\nabla_\mu\nabla_\nu f = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu\left(\sqrt{|\det g_{\alpha\beta}|} g^{\mu\nu}\partial_\nu f\right)$$

c) Show that for a vector field U^α, we have

$$\nabla_\mu U^\mu = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu\left(\sqrt{|\det g_{\alpha\beta}|} U^\mu\right) .$$

Similarly, for an anti-symmetric tensor $F^{\alpha\beta}$, show that

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu\left(\sqrt{|\det g_{\alpha\beta}|} F^{\mu\nu}\right) .$$

Does this remain true for totally-antisymmetric tensors with more indices?

2 Integral curves a) Given a vector field X^μ, recall that its integral curves are defined as the solutions of the equations

$$\frac{dx^\mu}{d\lambda} = X^\mu(x(\lambda)) .$$

Find the integral curves of the following vector fields on $\mathbb{R}^2$: ∂_x, $x\partial_y + y\partial_x$, $x\partial_y - y\partial_x$, $x\partial_x + y\partial_y$.

b) Let f be a function satisfying

$$g(\nabla f, \nabla f) = \psi(f) ,$$

for some function ψ. Let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; thus $d\gamma^\mu/d\lambda = \nabla^\mu f$. Find a reparameterization $s \mapsto \gamma(\lambda(s))$ of γ so that

$$\frac{D}{ds}\frac{d\gamma^\mu}{ds} = 0 .$$

c) Recall that, for the Schwarzschild metric, we may define the Eddington–Finkelstein coordinate v by

$$dv = dt + \frac{r}{r-2m}dr.$$

Show that, in the coordinates (v, r, θ, φ), the integral curves of the vector field ∇r meeting $\{r = 2m\}$ are null geodesics.

d) Let f be one of the coordinates, say $f = x^1$, in a coordinate system $\{x^i\}$. Verify that

$$g(\nabla f, \nabla f) = g^{11} \ .$$

Using this observation find a family of spacelike geodesics in the (t, r, θ, φ) coordinate system, as well as two distinct families of geodesics in the (v, r, θ, φ) coordinate system. Do any members of the second family coincide with members of the first?

3 [For self-study, will most likely not be covered in class.] Repeat part 1 of Question 1 with a connection ∇ that is metric (i.e. $\nabla_\alpha g_{\beta\gamma} = 0$) but has torsion (i.e. $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ non-zero). Show that, for such a connection, the first Bianchi identity takes the form

$$R^\delta{}_{[\alpha\beta\gamma]} = \nabla_{[\alpha} T^\delta{}_{\beta\gamma]} - T^\epsilon{}_{[\alpha\beta} T^\delta{}_{\gamma]\epsilon}.$$

[Hint: One way is to let ϕ be an arbitrary function and start from the identity $-R^\delta{}_{\gamma\alpha\beta}\nabla_\delta\phi = \nabla_\alpha\nabla_\beta\nabla_\gamma\phi - \nabla_\beta\nabla_\alpha\nabla_\gamma\phi + T^\delta{}_{\alpha\beta}\nabla_\delta\nabla_\gamma\phi$. Rewrite the second term using $\nabla_\alpha\nabla_\gamma\phi = \nabla_\gamma\nabla_\alpha\phi - T^\delta{}_{\alpha\gamma}\nabla_\delta\phi$. Then skew-symmetrise over α, β, γ. Manipulating what you get and stripping off the $\nabla\phi$ terms (using the fact that ϕ is arbitrary) should give you the desired result.]