1 Schwarzschild: Orders of magnitude Recall that the Schwarzschild metric g takes the form

$$g = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \ . \qquad (1)$$

Calculate the deviation of the component g_{00} of a Schwarzschild metric from minus one, as well as the deviation of $\partial_i g_{00}$ from zero, a) at the surface of the sun when m is the mass of the sun, b) at the orbit of the earth when m is the mass of the sun, c) at the surface of the earth when m is the mass of the earth, d) at the orbit of the moon when m is the mass of the earth, and e) at the surface of the moon when m is the mass of the moon.

Clearly, something wrong is happening with (1) at $r = 2m$. The corresponding radius is called the *S*chwarzschild radius. Calculate the Schwarzschild radius of a) the sun, b) the earth, and c) the moon. Calculate the corresponding mass densities, compare the result to the density of a neutron.

2 Symmetries of the curvature tensor

i. What does it mean for a connection ∇ on a space-time with metric g_{ab} to be (a) a *metric connection*, (b) *torsion free*?

Assume henceforth that ∇ is torsion free.

(c) Given an arbitrary smooth covector field A_a, and a smooth antisymmetric tensor field F_{ab}, show that

$$H_{ab} := \nabla_a A_b - \nabla_b A_a \quad \text{and} \quad \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab}$$

are both independent of the choice of the connection.

(d) Hence or otherwise show that

$$\nabla_a H_{bc} + \nabla_b H_{ca} + \nabla_c H_{ab} = 0 \ .$$

Recall that the curvature tensor is defined as

$$\nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c = R^c{}_{dab} V^d \ .$$

(e) Show that

$$\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = -R^d{}_{cab} A_d \ .$$

(f) Hence show that $R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab} = 0$ for a torsion-free connection.

(g) Show further that, for a tensor T_{ab},

$$\nabla_a \nabla_b T_{cd} - \nabla_b \nabla_a T_{cd} = -R^e{}_{cab} T_{ed} - R^e{}_{dab} T_{ce} \ .$$

(h) Hence show that $R_{abcd} = -R_{abdc}$ if ∇ is metric and torsion-free.

ii. Show that the symmetry $R_{abcd} = R_{cdab}$ follows from $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$ and $R_{[abc]d} = 0$.

3 Counting components

(1) In four dimensions, a tensor satisfies $T_{abcde} = T_{[abcde]}$. Show that $T_{abcde} = 0$.

(2) A tensor T_{ab} is *symmetric* if $T_{ab} = T_{(ab)}$. In n-dimensional space, it has n^2 components, but only $\frac{1}{2}n(n+1)$ of these can be specified independently—for example the components T_{ab} for $a \leq b$. How many independent components do the following tensors have (in n dimensions)?

(a) F_{ab} with $F_{ab} = F_{[ab]}$.
(b) A tensor of type $(0, k)$ such that $T_{ab\ldots c} = T_{[ab\ldots c]}$ (distinguish the cases $k \leq n$ and $k > n$, bearing in mind the result of question (1)).
(c) R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$.
(d) R_{abcd} with $R_{abcd} = R_{[ab]cd} = R_{ab[cd]} = R_{cdab}$.

Show that, in four dimensions, a tensor with the symmetries of the Riemann tensor has 20 independent components.