Übungen zur Vorlesung Relativitätstheorie und Kosmologie II: Revision Problem Sheet

1 Changes of coordinates

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients $\tilde{g}_{ab}$, and the components of X and α, in the coordinate system

$$\tilde{x}^0 = \tau, \quad \tilde{x}^1 = \phi, \quad \tilde{x}^2 = y, \quad \tilde{x}^3 = z,$$

where τ, ϕ are 'Rindler coordinates', defined by $t = \tau \cosh\phi$, $x = \tau \sinh\phi$. Determine which region of Minkowski space the coordinate system covers. *[Hint: A quick method for the metric is to write it as* $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ *and substitute, for example,* $\mathrm{d}x = \sinh\phi\, \mathrm{d}\tau + \tau\cosh\phi\, \mathrm{d}\phi$*, and so on.]*

2 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a\partial_a$ acting on functions as $X(f) = X^a\partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \; .$$

Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \; .$$

3 Christoffel symbols

The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \; .$$

i. Establish the transformation law

$$\Gamma^a_{bc} = \tilde{\Gamma}^d_{ef} \frac{\partial x^a}{\partial \tilde{x}^d} \frac{\partial \tilde{x}^e}{\partial x^b} \frac{\partial \tilde{x}^f}{\partial x^c} + \frac{\partial x^a}{\partial \tilde{x}^d} \frac{\partial^2 \tilde{x}^d}{\partial x^b \partial x^c}$$

by direct calculation. Explain why this implies that the Christoffel symbols do *not* define a tensor.

ii. Show that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{1}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b \tag{2}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0 \ . \tag{3}$$

iii. Show that if the metric does not explicitly depend upon a coordinate, say x^1, then $g(\dot{x}, \partial_1)$ is constant along every geodesic.

iv. Using the above variational principle for geodesics, write down the geodesic equations, and give the obvious constants of motion, for a metric of the form

$$-e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2 d\varphi^2 \ . \tag{4}$$

v. Use (1) to calculate the Christoffel symbols for the metric (4).