1 Alice circles planet X freely for a long time on a circular orbit of radius R, while her twin Bob remains motionless on the surface of the planet, at radius r_0. For $r \geq r_0$ the geometry of the gravitational field of the planet X is described by the Schwarzschild metric with mass $0 < m < r_0/2$. Derive a necessary and sufficient condition on R which guarantees that, on meeting Bob again, Alice will have the same age as Bob. You should assume that the time of travel back and forth from radius R to radius r_0 can be neglected compared to the time that Alice spent on the circular orbit.

2 Recall that (PS3 Q 1)

$$\nabla^\mu \nabla_\mu f = \frac{1}{\sqrt{|\det g_{\alpha\beta}|}} \partial_\mu \left(\sqrt{|\det g_{\alpha\beta}|} g^{\mu\nu} \partial_\nu f \right) .$$

i. Show that, for weak gravitation fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ appropriately small, the *wave coordinates* condition

$$\nabla^\mu \nabla_\mu x^\alpha = 0$$

approximately reads

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu (h^\alpha{}_\alpha) ,$$

where $h^\alpha{}_\beta = \eta^{\alpha\gamma} h_{\gamma\beta}$.

ii. Let $h_{\alpha\beta} = A_{\alpha\beta} \cos(k_\mu x^\mu)$ be a linearized gravitational wave in TT gauge. Show that

$$R_{\alpha\beta\gamma\delta} k^\delta = 0 .$$