1 A vector field X^μ is called *Killing* if

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \; .$$

i. Check that ∂_t and ∂_φ are Killing vectors both in the Minkowski metric and in the Schwarzschild metric.

ii. Show that if γ is a geodesic and X is Killing, then $g(\gamma, X)$ is constant along γ.

iii. Show that if $\nabla_\mu T^\mu{}_\nu = 0$ and X is Killing then the vector field $J^\mu := T^\mu{}_\nu X^\nu$ has vanishing divergence, $\nabla_\mu J^\mu = 0$.

2 Let φ satisfy the wave equation in a general space-time with Lorentzian metric g, $\Box_g \varphi := \nabla^\mu \nabla_\mu \varphi = 0$. Set

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \nabla^\alpha \varphi \partial_\alpha \varphi g_{\mu\nu} \; .$$

Show that $\nabla_\mu T^\mu{}_\nu = 0$.

Show that T satisfies the *dominant energy condition*: $T_{\mu\nu} X^\mu Y^\nu \geq 0$ for all timelike future directed X and Y. Is this condition satisfied for the energy-momentum tensor of dust?

When $X = \partial_t$ is a Killing vector, the constant of motion associated with the current J of question 1iii is called the total energy E of the field. Give an explicit expression for E for the surface $\{t = 0\}$ in the Minkowski space-time, and in the Schwarzschild space-time.

Assuming that $g = \eta$, the Minkowski metric, and that $\mathscr{S}_u$ are the family of hypersurfaces asymptotic, when $r \to \infty$, to the light cones $t = r + u$, as in the lectures, derive a formula for the rate of change $\frac{d}{du} p_i(\mathscr{S}_u)$ of the total space-momentum $\vec{p}(\mathscr{S}_u)$ of $\mathscr{S}_u$.

3 Calculate the field of unit normals and the induced metric for

i. S^2 included in a flat $\mathbb{R}^3$ (use polar coordinates on S^2)

ii. S^2 viewed as a sphere of constant radius in Schwarzschild (use polar coordinates on S^2)

iii. $\mathcal{S} = \{x^0 = \sqrt{1 + r^2}\}$ in four dimensional Minkowski; use the cartesian space coordinates x^i as coordinates on $\mathcal{S}$

iv. $\mathcal{S}' = \{r = \sqrt{1 + t^2}\}$ in four dimensional Minkowski; use t and polar coordinates as coordinates on $\mathcal{S}'$

4 Find a function f so that the metric induced on the hypersurface $\{t = f(r)\}$ in Schwarzschild space-time is flat.