1 Suppose that the spatial volume of a closed, matter dominated, FRW universe is 10^{12}Mpc^3 at the moment of maximum expansion. What is the duration of this universe from big bang to big crunch in years?

2 Recall that for FRW models with any combination of matter and radiation and with non-positive cosmological constant the scale factor R is a concave function of t (i.e. $\frac{d^2R}{dt^2} \leq 0$). Assuming that $R(t) \sim t$ as $t \to 0$, deduce from this that $1/H(t) \geq t$ for $t > 0$.

3 Given a Lorentzian metric g, we define the *Einstein–Hilbert action* to be the integral of the scalar curvature of the metric over the manifold

$$S_{EH}[g] = \int_M R \, dV_g \equiv \int_M R \sqrt{-\det g} \, d^n x.$$

We wish to find critical points of this action when we consider varying the metric g. As such, we consider a one-parameter family of metrics, $g(t)$, with $g(0) = g$, and define the derivative

$$h_{\alpha\beta} := \frac{d}{dt} g_{\alpha\beta}(t)\Big|_{t=0}.$$

Our aim is to calculate $\frac{d}{dt} S_{EH}[g(t)]\Big|_{t=0}$.

i. Defining $g^{\alpha\beta}(t)$ such that $g^{\alpha\beta}(t) g_{\beta\gamma}(t) = \delta^\alpha_\gamma$ show that

$$\frac{d}{dt} g^{\alpha\beta}(t)\Big|_{t=0} = -g^{\alpha\gamma} h_{\gamma\delta} g^{\beta\delta}.$$

ii. Show that

$$\frac{d}{dt} \sqrt{-\det g(t)}\Big|_{t=0} = \frac{1}{2} \sqrt{-\det g} \, g^{\alpha\beta} h_{\alpha\beta}.$$

iii. Using the formula for the Christoffel symbols of the metric $g(t)$, show that

$$\dot{\Gamma}^\alpha{}_{\beta\gamma} := \frac{d}{dt} \Gamma^\alpha{}_{\beta\gamma}[g(t)]\Big|_{t=0} = \frac{1}{2} g^{\alpha\delta} \left[\nabla_\beta h_{\gamma\delta} + \nabla_\gamma h_{\beta\delta} - \nabla_\delta h_{\beta\gamma} \right].$$

[Hint: it is best to use a coordinate system around the point p such that the derivatives of the metric vanish at p, and to notice that $\dot{\Gamma}^\alpha{}_{\beta\gamma}$ is a tensor (why?).]

iv. Finally, using the formula

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma{}_{\alpha\beta} - \partial_\alpha \Gamma^\gamma{}_{\gamma\beta} + \Gamma^\gamma_{\gamma\delta} \Gamma^\delta{}_{\alpha\beta} - \Gamma^\gamma_{\alpha\delta} \Gamma^\delta{}_{\gamma\beta},$$

deduce that

$$\frac{d}{dt} R_{\alpha\beta}[g(t)]\Big|_{t=0} = \nabla_\gamma \dot{\Gamma}^\gamma{}_{\alpha\beta} - \nabla_\alpha \dot{\Gamma}^\gamma{}_{\gamma\beta}$$

[Hint: use again a coordinate system around p such that the derivatives of the metric vanish at p.] and, therefore, that

$$g^{\alpha\beta}\,\frac{d}{dt}R_{\alpha\beta}[g(t)]\Big|_{t=0} = \nabla_\gamma V^\gamma,$$

where V is the vector field with components

$$V^\gamma = g^{\alpha\beta}\dot{\Gamma}^\gamma{}_{\alpha\beta} - g^{\gamma\beta}\dot{\Gamma}^\alpha{}_{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta}\left[\nabla_\beta h_{\alpha\delta} + \nabla_\alpha h_{\beta\delta} - 2\nabla_\delta h_{\alpha\beta}\right].$$

Noting that $R[g(t)] = g^{\alpha\beta}(t)R_{\alpha\beta}[g(t)]$, show that

$$\frac{d}{dt}S_{EH}[g(t)]\Big|_{t=0} = \int_M \left(-R^{\alpha\beta}[g] + \frac{1}{2}R[g]g^{\alpha\beta}\right)h_{\alpha\beta}\,dV_g + \int_M \nabla_\gamma V^\gamma\,dV_g.$$

Using Stokes's theorem, deduce that the variation of the Einstein–Hilbert action takes the form

$$\frac{d}{dt}S_{EH}[g(t)]\Big|_{t=0} = -\int_M G^{\alpha\beta}[g]h_{\alpha\beta}\,dV_g.$$

A critical point of the Einstein–Hilbert action is a metric g for which $\frac{d}{dt}S_{EH}[g(t)]\big|_{t=0} = 0$ for all compactly supported perturbations $g(t)$ with $g(0) = g$. Deduce from the previous formula that such a critical point of the action is necessarily a solution of the vacuum Einstein equations.