1 The structure constants ${C^i}_{jk}$ of a Lie algebra spanned on vector fields X_i, $i = 1, \ldots, N$, are defined by the formula

$$[X_i, X_j] = {C^k}_{ij} X_k \ .$$

Write the equation satisfied by the ${C^i}_{jk}$'s that follows from Jacobi's identity.

Find the structure constants for the Heisenberg group (see Q3 of PS11).

The Lie algebra of $SO(3)$ is generated by $X_i = \epsilon_{ijk} x^j \partial_k$, $i = 1, 2, 3$; why? *[Hint: the flow of X_3 has been calculated in Q1 of PS11].* Find the structure constants of that Lie algebra.

2 Find the geodesics of the maximally symmetric Riemannian and Lorentzian manifolds. *[Hint: it suffices to find a family with the property that all geodesics can be obtained from the members of the family by applying isometries.]*

Using this, or otherwise, show that in the embedded model where the maximally symmetric manifold is a submanifold $\mathcal{S}_a$ of $\mathbb{R}^3$ as described in the lectures, the geodesics are intersections of $\mathcal{S}_a$ with planes through the origin.

3 Let G be a matrix Lie group (i.e. G is a subset of the group of real or complex, $n \times n$ matrices, containing the identity matrix, 1, and satisfying the properties of a Lie group, with group multiplication given by matrix multiplication). Since any tangent vector to any manifold can be represented as $d\gamma(t)/dt|_{t=0}$, where γ is a curve on the manifold, both points in G and vectors tangent to G can be represented by matrices. A vector field X on G can thus be viewed as the assignment of a matrix $X(g)$ to every matrix $g \in G$.

Let $A \in T_1 G$ be a tangent vector at the identity (so that $A = \dot{\gamma}(0)$, where γ is a smooth curve in G with $\gamma(0) = 1$.) For $g \in G$, one defines $L_g : G \to G$ by the formula $L_g h := gh$ (hence, $\left(L_g\right)_* A \in T_g G$ is the vector $\frac{d}{dt}\left(g\gamma(t)\right)\Big|_{t=0}$).

Show that the matrix representing the left-invariant vector field which equals A at the identity, say X_A, at $g \in G$ equals

$$X_A(g) = gA, \qquad g \in G.$$

Show that the integral curve of this vector field, $\phi(t) \in G$, with initial point $\phi(0) = 1$ is given by

$$\phi(t) = e^{tA}.$$

Using uniqueness of solutions of ordinary differential equations, or otherwise, show that

$$\phi(s + t) = \phi(s)\,\phi(t) = \phi(t)\,\phi(s).$$

[Hint: Let $f(t) := \phi(t)\phi(s)$, and $g(t) = \phi(t + s)$. Show that $\frac{d}{dt} f(t) = Af(t)$ with $f(0) = e^{sA}$ and $\frac{d}{dt} g(t) = Ag(t)$ with $g(0) = e^{sA}$.]

Show that the integral curve of the left-invariant vector field X_A such that $X_A(1) = A$, with initial point $g_0 \in G$ is given by

$$g(t) = g_0\, e^{tA}.$$

Conclude that X_A is complete, and that its flow $\phi_t[X_A]$ is given by

$$\phi_t[X_A](g) = ge^{tA}\ .$$

Show that the commutator of two left-invariant vector fields X_A and X_B is the vector field $X_{[A,B]}$, where $[A, B]$ is the usual commutator of matrices. *[Hint: let f be any function on G, check that at $g \in G$ it holds that*

$$[X_A, X_B](f)(g) = \frac{d^2(f(ge^{tA}e^{sB}))}{dt\,ds}\Big|_{t=s=0} - \frac{d^2(f(ge^{sB}e^{tA}))}{dt\,ds}\Big|_{t=s=0}\ ;$$

conclude by making a Taylor expansion in t and s to calculate the right-hand side.]