1 Derive the red-shift formula for radial photons moving in the metric

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \ . \tag{1}$$

Derive the force that replaces the Newtonian gravitational force acting on test bodies moving in this gravitational field for large r and for small velocities with respect to the static observers.

2 Consider a Newtonian body of mass m_0 and angular momentum J_N moving on a bound Keplerian orbit in the Newtonian field of a spherically symmetric body with mass m. Set $u = m/r$. Show that

$$u = \frac{G m_0^2 m}{J_N^2}(e \cos\varphi + 1) \ ,$$

where e is the eccentricity of the orbit.

Assuming a circular orbit, calculate m/r, J_N/r^2, m/J_N^2 for a) the orbit of the earth around the sun, and for b) the orbit of Mercury around the sun. Estimate the rate of precession of the perihelion of the orbit of the earth, and of that of Mercury.

3 a) Let $u = m/r$. Show that there exist constants E, J and λ such that along nonradial geodesics in the Schwarzschild geometry we have

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{m^2 E^2}{J^2} - (u^2 + \frac{\lambda m^2}{J^2})(1 - 2u) \ . \tag{2}$$

b) Show that for every $r > 3m$ there exist timelike geodesics for which $\dot{r} = 0$.

c) Consider a geodesic which is a small perturbation of the fixed-radius geodesic of point b). Writing $u = u_0 + \delta u$, where $du_0/d\varphi = 0$, and where δu is assumed to be small, derive a linear second order differential equation approximatively satisfied by δu. Solving this equation, conclude that for $3m < r < 6m$ the radial geodesics are unstable at a linearized level, while they are linearization-stable for $r > 6m$.

4 Find the equivalent of (2) for geodesics for the metric (1).