1 Integral curves of gradients 1. Let f be a function satisfying

$$g(\nabla f, \nabla f) = \psi(f) \ ,$$

for some function ψ. Let $\lambda \mapsto \gamma(\lambda)$ be any integral curve of ∇f; by definition, this means that $d\gamma^\mu/d\lambda = \nabla^\mu f$. Find a reparameterization $s \mapsto \gamma(\lambda(s))$ of γ so that

$$\frac{D}{ds}\frac{d\gamma^\mu}{ds} = 0 \ .$$

Conclude that, in the Eddington-Finkelstein coordinates (v, r, θ, φ), the integral curves of ∇r meeting $\{r = 2m\}$ are null geodesics.

2. Let f be one of the coordinates, say $f = x^1$, in a coordinate system $\{x^i\}$. Verify that

$$g(\nabla f, \nabla f) = g^{11} \ .$$

Using this observation find a family of spacelike geodesics in the (t, r, θ, φ) coordinate system, as well as two distinct families of geodesics in the (v, r, θ, φ) coordinate system. Do any members of the second family coincide with members of the first?

2 A non-physical black hole metric. Let a be a strictly positive constant, and for $r > a$ let g be a Lorentzian metric of the form

$$g = -\left(1 - \frac{a}{r}\right)^3 dt^2 + \frac{dr^2}{\left(1 - \frac{a}{r}\right)^3} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \ . \tag{1}$$

1. Proceeding in a way analogous to the analysis of the Schwarzschild metric, replace t by a new coordinate v so that the metric, in the new coordinates, can be smoothly extended from the original manifold $\{t \in \mathbb{R}\} \times \{r > a\} \times S^2$ to a Lorentzian metric on a new manifold $\{v \in \mathbb{R}\} \times \{r > 0\} \times S^2$.

2. Show that, with an appropriate choice of the coordinate v, the set $\{r < a\}$ in the extended manifold is a black hole region.

3. Find the four-acceleration of stationary observers for this metric.

4. How would you show that $\{r = 0\}$ is a singular set for the metric (1)? For the ambitious: use an algebraic manipulation program to carry this out.