You are expected to cross three problems amongst exercises 37, 38, 39 and 40. As usual, you are strongly encouraged to attempt more than three.

36 **("Raising and lowering of indices")** Define

$$B_\alpha := \eta_{\alpha\beta} A^\beta \ , \quad C^\gamma := \eta^{\gamma\sigma} B_\sigma \ . \tag{1}$$

Show that

$$C^\gamma = A^\gamma \ . \tag{2}$$

The first operation in (1) is called "lowering an index with the metric"; the second "raising an index with the metric". What does (2) say about these operations?

From now on we shall simply write

$$A_\alpha := \eta_{\alpha\beta} A^\beta \ , \quad B^\gamma := \eta^{\gamma\sigma} B_\sigma \ .$$

Show that

$$A_\alpha B^\alpha = A^\alpha B_\alpha \ .$$

37 Define a $*$-operation on anti-symmetric tensors as

$$*F_{\alpha\beta} := \frac{1}{2} \epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} \ , \qquad *F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \ ,$$

where

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} := \eta_{\alpha\mu} \eta_{\beta\nu} \epsilon^{\mu\nu\gamma\delta} \ .$$

i. Show that

$$\partial_\mu (*F^{\mu\nu}) = 0 \quad \Longleftrightarrow \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \partial_{[\alpha} F_{\beta\gamma]} = 0 \, .$$

ii. Show that

$$*F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \ .$$

iii. Show that the double star of an anti-symmetric tensor is the negative of this tensor.

iv. Show that if $F_{\mu\nu}$ and $G_{\mu\nu}$ are anti-symmetric, then $*F^{\alpha\beta} G_{\alpha\beta} = F^{\alpha\beta} *G_{\alpha\beta}$. Conclude that $*F^{\alpha\beta} *F_{\alpha\beta} = -F^{\alpha\beta} F_{\alpha\beta}$. Can you think of a simpler proof of the last equality?

38 Consider a charged particle which moves along *a straight line in Minkowski space-time* in an electromagnetic field. Show that its velocity $\vec{v}$ satisfies $\vec{E} + \vec{v} \times \vec{B} = 0$ and $\vec{E} \cdot \vec{v} = 0$. Find all possible solutions for $\vec{v}$ in terms of $\vec{E}$ and $\vec{B}$. What can you say about $F^{\alpha\beta} F_{\alpha\beta}$ and $*F^{\alpha\beta} F_{\alpha\beta}$?

39 The tensor field

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\eta_{\mu\nu}\right) \tag{3}$$

is called the *energy-momentum tensor* of the electromagnetic field. Express T_{00}, T_{0i}, and T_{ij} in terms of E^i and B^j.

40 Describe the gauge transformations which preserve the Lorenz gauge condition.

41 [For self study:] Let $T_{\mu\nu}$ be given by (3). Show that

$$T_{\mu\rho}T^{\rho}{}_{\nu} = \frac{1}{4}T_{\alpha\beta}T^{\alpha\beta}\eta_{\mu\nu} \ .$$

[Hint: an efficient proof uses the Cayley-Hamilton theorem.]