1 Let a bracket over indices denote complete antisymmetrisation, and a parenthesis over indices denote complete symmetrisation: for example,

$$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\mu\nu}) \ , \qquad A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\mu\nu}) \ , \quad \delta_\mu^{[\alpha}\delta_\rho^{\gamma]} = \frac{1}{2}(\delta_\mu^\alpha\delta_\rho^\gamma - \delta_\mu^\gamma\delta_\rho^\alpha) \ ,$$

and similarly for 3, 4 or more indeces. Show that

i. $A^{[\mu\nu]}B_{\mu\nu} = A^{\mu\nu}B_{[\mu\nu]}$,

ii. $A^{[\mu\nu\rho]}B_{\mu\nu\rho} = A^{\mu\nu\rho}B_{[\mu\nu\rho]}$,

iii. $\delta_\mu^{[\alpha}\delta_\rho^{\gamma]} = \delta_{[\mu}^\alpha\delta_{\rho]}^\gamma$,

iv. $\delta_\mu^{[\alpha}\delta_\nu^\beta\delta_\rho^{\gamma]} = \delta_{[\mu}^\alpha\delta_\nu^\beta\delta_{\rho]}^\gamma$,

v. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\rho} = -6\delta_\rho^\delta$,

vi. $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\nu\rho} = -4\delta_{[\nu}^\gamma\delta_{\rho]}^\delta$.

2 Assuming the tensorial transformation law of $F^{\mu\nu}$, derive the explicit formulae for the transformation laws of the electric and magnetic fields under a boost along the first coordinate axis.

3 1) Assuming that $\Lambda^\alpha{}_\beta$ is a Lorentz matrix, show that $\eta^{\alpha\beta}\Lambda^\mu{}_\beta\eta_{\mu\nu}$ is inverse to $\Lambda^\alpha{}_\beta$.

2) Recall that we required that $F^{\mu\nu}$ transforms as follows under Lorentz transformations: if $\bar{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$, then

$$\overline{F}^{\mu\nu}(\bar{x}) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}(x) \ ,$$

and that $F_{\mu\nu}$ has been defined as

$$F_{\mu\nu} := \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta} \ .$$

Use 1) to show that $F_{\alpha\beta}$ transforms as

$$\overline{F}_{\mu\nu}(\bar{x}) = (\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu F_{\alpha\beta}(x)$$

(this is called the *transformation law of a two-covariant tensor*);

4 Show that contractions $F_{\alpha\beta}F^{\alpha\beta}$ and $F^{\alpha\beta} * F_{\alpha\beta}$ are invariant (more precisely, behave as scalars) under Lorentz transformations. Express them in terms of $\vec{E}$ and $\vec{B}$.

5 Let $\vec{E}\cdot\vec{B} = 0$, and suppose that $|\vec{E}|^2 \neq |\vec{B}|^2$. Show that there exists a Lorentz frame in which either $\vec{E}$ or $\vec{B}$ vanishes.