1 Lie bracket

Recall that vector fields can be identified with homogeneous linear first order partial differential operators $X = X^a \partial_a$ acting on functions as $X(f) = X^a \partial_a f$.

The Lie-bracket $[X, Y]$ of two vector fields X and Y is defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Show that $[X, Y]$ also is a vector field, i.e. a homogeneous linear first order differential operator, with components

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a . \tag{1}$$

Check, by a direct coordinate calculation, that the right-hand-side of (1) transforms as a vector field under changes of coordinates.

[For self-study:] Prove the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$

2 Christoffel symbols

The Christoffel symbols of the Levi-Civita connection are defined by the formula

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) .$$

Using this definition, calculate the Christoffel symbols for a) the Euclidean metric on $\mathbb{R}^2$ in polar coordinates: $g = d\rho^2 + \rho^2 d\varphi^2$, and b) the unit round metric on S^2: $h = d\theta^2 + \sin^2\theta d\varphi^2$.

Show that the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \tag{2}$$

associated with the Lagrange function

$$L(x^c, \dot{x}^c) = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b \tag{3}$$

can be written as

$$\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0 . \tag{4}$$

Use the explicit form of (2) for the two-dimensional metrics g and h above to calculate again the Christoffel symbols.

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3 Geodesics

Solutions $x^a(t)$ of (4) are called *affinely parameterized geodesics*, and the parameter t is said to be *affine*.

i. For a vector field X defined along a curve $s \mapsto \gamma(s)$, set

$$\frac{DX^a}{ds} := \frac{dX^a}{ds} + \Gamma^a{}_{bc}\dot{\gamma}^b X^c \ ,$$

where the $\Gamma^a{}_{bc}$'s are the Christoffel symbols of the Levi-Civita connection associated with the metric g. Show that

$$\frac{d(g(X,Y)\circ\gamma)}{ds} = g(\frac{DX}{Ds}, Y) + g(X, \frac{DY}{Ds}) \ .$$

ii. Let $x^a(\sigma)$ be a geodesic with an affine parameter σ and let $W^a = \mathrm{d}x^a/\mathrm{d}\sigma$. Show that:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(g_{ab}W^a W^b) = 0.$$

Recall that a vector X is called *timelike* if $g(X,X) > 0$, *null* if $X \neq 0$ and $g(X,X) = 0$, and *spacelike* if $g(X,X) < 0$. Deduce that the type of the tangent vector does not change along a geodesic. Show that for a timelike geodesic, σ is an affine function of proper time τ (this means that $\sigma = a\tau + b$, for some constants a, b).