

1 Newtonian warm-up

An elevator shaft is drilled diametrically through an ideally homogeneous spherical earth of radius R and density ρ . An elevator is dropped into this shaft from rest at the surface. Two point particles A and A' , are initially at rest in the elevator, one at the center of the ceiling and one at the center of the floor, distance d apart. (i) Show that the whole cabin executes simple harmonic motion with respect to the shaft, the period of which you will determine. (ii) Find the relative acceleration between the particles in the frame defined by the center of mass of the elevator. (This relative acceleration is often referred to as “tidal gravitational force”). [Hint: Use Gauss’ theorem, equating influx of the gravitational field through a given surface to $4\pi G$ the enclosed mass.]

2 Changes of coordinates

Let $x^0 = t, x^1 = x, x^2 = y, x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let X be the vector field which in the above coordinate system equals $(1, 1, 0, 0)$, and let α be a one-form which in the above coordinate system equals $(1, 1, 0, 0)$.

Find the metric coefficients \tilde{g}_{ab} , and the components of X and α , in each of the following coordinate systems.

- (i) $\tilde{x}^0 = t - z, \tilde{x}^1 = r, \tilde{x}^2 = \theta, \tilde{x}^3 = z$
- (ii) $\tilde{x}^0 = \tau, \tilde{x}^1 = \phi, \tilde{x}^2 = y, \tilde{x}^3 = z,$
- (iii) $\tilde{x}^0 = t, \tilde{x}^1 = r, \tilde{x}^2 = \theta, \tilde{x}^3 = \phi$

where, in the first case, r, θ are plane polar coordinates in the x, y plane; in the second, τ, ϕ are ‘Rindler coordinates’, defined by $t = \tau \cosh \phi, x = \tau \sinh \phi$; and, in the third, r, θ, ϕ are spherical polar coordinates. In each case, state which region of Minkowski space the coordinate system covers. [Hint: A quick method for the metric is to write it as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ and substitute, for example, $dx = \cos \theta dr - r \sin \theta d\theta$, and so on. Of course you should convince yourself that this is legitimate.]