BASIC NOTES ON WAVE EQUATIONS

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These notes are a review of basic material on wave equations and the initial value problem. Section 1 is essentially a summary of the material contained in pp. 65–69 of Evans [1] (see, also, Chapter 1 of Sogge [4]). Most of the material in Section 2 in, for example, Chapters 3.3–3.5 of [5]. For more information on Schwartz spaces, tempered distributions, etc, see, for instance, Chapter 5 of [2]. A good introduction to all of this material (and much more) are the recent lecture notes by Klainerman [3].

1. The wave equation on Minkowski space

We want to solve the (inhomogeneous) wave equation

$$u_{tt} - \Delta u = f, \tag{1.1}$$

subject to appropriate boundary conditions and initial conditions. Here $t > 0$ and $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial {x^i}^2}$ where $\mathbf{x} := (x^1, \dots, x^n)$ lies in $U \subseteq \mathbb{R}^n$, an open subset of $\mathbb{R}^n$. Throughout, we will use subscripts to denote partial derivatives, so

$$u_t \equiv \frac{\partial u}{\partial t}, \qquad u_{tx} \equiv \frac{\partial^2 u}{\partial t\, \partial x}, \qquad \text{etc.}$$

Given a function $f \colon (0, \infty) \times U \to \mathbb{R}$, we view (1.1) (along with boundary conditions introduced below) as an equation for the unknown $u \colon [0, \infty) \times \overline{U} \to \mathbb{R}$, i.e. $u(t, \mathbf{x})$. In the case $U = \mathbb{R}^n$, appropriate boundary conditions would be to impose that

$$u(0, \mathbf{x}) = g(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = h(\mathbf{x}),$$

where g, h are (for example) smooth functions with compact support on $\mathbb{R}^n$.[1] We define the wave operator

$$\Box := \partial_t^2 - \Delta,$$

in terms of which the wave equation takes the form

$$\Box u = f. \tag{1.2}$$

We will often simplify to the homogeneous problem with $f = 0$, i.e.

$$\Box u = 0, \tag{1.3}$$

again subject to appropriate boundary conditions.

1.1. $n = 1$.

In the case $n = 1$, taking $U = \mathbb{R}$, we consider the homogeneous wave equation

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}$$

with initial conditions

$$u(0, x) = g(x), \qquad u_t(0, x) = h(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

Proposition 1.1. *There exists a unique solution $u(t, x)$ of this problem, given by the* d'Alembert formula

$$u(t, x) = \frac{1}{2} \left[g(x + t) + g(x - t) \right] + \frac{1}{2} \int_{x-t}^{x+t} h(s)\, ds. \tag{1.4}$$

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[1] We will not be concerned with optimal regularity issues in this course, and will assume that all data such as f, g, h are as well-behaved as necessary for our arguments to be classically valid.

Proof of existence. To prove existence, we simply show that the d'Alembert formula gives a function u with all of the required properties. It follows directly from (1.4) that u satisfies the required boundary. Now, note that we can write

$$u(t,x) = F(t+x) + G(t-x)$$

where

$$F(u) = \frac{1}{2}g(u) + \frac{1}{2}\int_{u_0}^{u} h(s)\,ds, \qquad G(u) = \frac{1}{2}g(u) + \frac{1}{2}\int_{u}^{u_0} h(s),$$

where u_0 is any fixed real number. Since $F(t+x)$ and $G(t-x)$ separately satisfy the wave equation, it follows that u, as defined in (1.4), satisfies the wave equation. □

In the standard approach, the d'Alembert formula is constructed as the unique solution of the wave equation. Rather than pursue this course, uniqueness of the solution (1.4) will follow from the following, more general, discussion.

1.2. **Energy methods.**

1.2.1. *Uniqueness.* We return to the more general problem, assuming that $U \subset \mathbb{R}^n$ is a bounded, open set with smooth boundary ∂U. Given $T > 0$, we define the sets

$$U_T := (0,T] \times U$$

and

$$\Gamma_T := \overline{U_T} \backslash U_T = (\{0\} \times U) \cup ([0,T] \times \partial U).$$

Theorem 1.2. *Given functions $g\colon \Gamma_T \to \mathbb{R}$ and $h\colon U \to \mathbb{R}$, if there exists a solution of the following problem*

$$\Box u = 0 \quad \textit{in } U_T \tag{1.5a}$$

$$u = g \quad \textit{on } \Gamma_T \tag{1.5b}$$

$$u_t = h \quad \textit{on } \{0\} \times U, \tag{1.5c}$$

then this solution is unique.

Proof. Let $u, \tilde{u}$ be solutions of (1.5) and define $w := u - \tilde{u}$. It follows that w satisfies

$$\Box w = 0 \quad \text{in } U_T \tag{1.6a}$$

$$w = g \quad \text{on } \Gamma_T \tag{1.6b}$$

$$w_t = h \quad \text{on } \{0\} \times U, \tag{1.6c}$$

Define the energy

$$E(t) := \frac{1}{2}\int_U \left[w_t(t,\mathbf{x})^2 + |\boldsymbol{\nabla} w(t,\mathbf{x})|^2 \right] d\mathbf{x},$$

where $\boldsymbol{\nabla} := (\partial_{x^1}, \dots, \partial_{x^n})$ denotes the spatial gradient and

$$|\boldsymbol{\nabla} w|^2 = \sum_{i=1}^{n} \left(\frac{\partial w}{\partial x^i}\right)^2 \equiv w_i w_i,$$

employing the Einstein summation convention. We then have

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_U \left[w_t w_{tt} + w_i w_{ti} \right] d\mathbf{x} \\
&= \int_U \left[w_t w_{tt} + \operatorname{div}(w_t \operatorname{grad} w) - w_t \operatorname{div}\operatorname{grad} w \right] d\mathbf{x} \\
&= \int_U w_t \left(w_{tt} - \Delta w \right) d\mathbf{x} + \int_{\partial U} w_t \operatorname{grad} w \cdot dS,
\end{aligned}$$

where we have used Stokes's theorem in the final line. The first term vanishes, since $w_{tt} - \Delta w = 0$. Also, since $w = 0$ on the set $[0, T] \times U$, it follows that $w_t = 0$ on this set. Therefore $w_t|\,\partial U = 0$, so the second integral also vanishes. Therefore

$$\frac{d}{dt}E(t) = 0,$$

so $E(t) = E(0)$. The boundary conditions state that $w_t(0, \mathbf{x}) = 0$. Moreover, $w(0, \mathbf{x}) = 0$ for all $\mathbf{x} \in U$, hence $\boldsymbol{\nabla} w(0, \mathbf{x}) = 0$. Therefore $E(0) = 0$ and thus $E(t) = 0$. It follows that $w_t = \boldsymbol{\nabla} w = 0$ on U_t, so w is constant. Since $w(0, \mathbf{x}) = 0$, it follows that $w = 0$ on U_T and therefore $\tilde{u} = u$. □

Remark 1.3. The same argument holds for the inhomogeneous problem, with $\Box u = f$ in (1.5).

Proof of uniqueness in Proposition 1.1. Taking $n = 1$ and $U = \mathbb{R}$, uniqueness of the d'Alembert solution follows from Theorem 1.2. □

1.2.2. *Domain of dependence.* In the d'Alembert formula (1.4), $u(t, x)$ depends only on the values of g at $x \pm t$ and on the values of h in the interval $[x - t, x + t]$. As such, the values of g and h outside of the past null cone of the point (t, x) have no influence on the value of the solution u at that point. This "finite speed of propagation" phenomenon is general feature of wave equations, which we can again analyse by energy methods.

Notation. Given $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, we denote by $B(\mathbf{x}, r)$ the open ball $\{\mathbf{y} \in \mathbb{R}^n \,|\, |\mathbf{y} - \mathbf{x}| < r\}$ and by $S(\mathbf{x}, r) \equiv \partial B(\mathbf{x}, r)$ the sphere $\{\mathbf{y} \in \mathbb{R}^n \,|\, |\mathbf{y} - \mathbf{x}| = r\}$. We will also consider the closed ball $\overline{B}(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^n \,|\, |\mathbf{y} - \mathbf{x}| \leqslant r\} = B(\mathbf{x}, r) \cup S(\mathbf{x}, r)$.

Let $t_0 > 0$ and $\mathbf{x}_0 \in \mathbb{R}^n$. We define the cone

$$C := \{(t, \mathbf{x}) \,|\, 0 \leqslant t \leqslant t_0, |\mathbf{x} - \mathbf{x}_0| \leqslant t_0 - t\}.$$

We then have the following finite-propagation speed result.

Proposition 1.4. *Let u be a solution of $\Box u = 0$ in $(0, \infty) \times \mathbb{R}^n$ with $u = u_t = 0$ on $\{0\} \times \overline{B}(\mathbf{x}_0, t_0)$. Then $u = 0$ on C.*

Proof. Let

$$E(t) := \frac{1}{2} \int_{B(\mathbf{x}_0, t_0 - t)} \left[u_t(t, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] d\mathbf{x}.$$

For $\epsilon > 0$, we then have

$$\begin{aligned}
\frac{E(t + \epsilon) - E(t)}{\epsilon} &= \frac{1}{2\epsilon} \left[\int_{B(\mathbf{x}_0, t_0 - t - \epsilon)} \left[u_t(t + \epsilon, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t + \epsilon, \mathbf{x})|^2\right] d\mathbf{x} \right. \\
&\qquad \left. - \int_{B(\mathbf{x}_0, t_0 - t)} \left[u_t(t, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] d\mathbf{x} \right] \\
&= \frac{1}{2\epsilon} \int_{B(\mathbf{x}_0, t_0 - t - \epsilon)} \left[u_t(t + \epsilon, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t + \epsilon, \mathbf{x})|^2 - u_t(t, \mathbf{x})^2 - |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] d\mathbf{x} \\
&\qquad - \frac{1}{2\epsilon} \int_{B(\mathbf{x}_0, t_0 - t) \setminus B(\mathbf{x}_0, t_0 - t - \epsilon)} \left[u_t(t, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] d\mathbf{x} \\
&= \frac{1}{2} \int_{B(\mathbf{x}_0, t_0 - t)} \partial_t \left[u_t(t, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] d\mathbf{x} + O(\epsilon) \\
&\qquad - \frac{1}{2\epsilon} \int_{t_0 - t - \epsilon}^{t_0 - t} \int_{S(\mathbf{x}_0, r)} \left[u_t(t, \mathbf{x})^2 + |\boldsymbol{\nabla} u(t, \mathbf{x})|^2\right] dS\, dr.
\end{aligned}$$

As $\epsilon \to 0$, we deduce that

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{B(\mathbf{x}_0, t_0 - t)} \partial_t \left[u_t u_{tt} + u_i u_{ti}\right] d\mathbf{x} - \frac{1}{2} \int_{S(\mathbf{x}_0, t_0 - t)} \left[u_t^2 + |\boldsymbol{\nabla} u|^2\right] dS \\
&= \int_{B(\mathbf{x}_0, t_0 - t)} \partial_t u_t \left[u_{tt} - \Delta u\right] d\mathbf{x} + \int_{S(\mathbf{x}_0, t_0 - t)} u_t \boldsymbol{\nabla} u \cdot \mathbf{n}\, dS - \frac{1}{2} \int_{S(\mathbf{x}_0, t_0 - t)} \left[u_t^2 + |\boldsymbol{\nabla} u|^2\right] dS,
\end{aligned}$$

where $\mathbf{n}$ denotes the unit normal to $S(\mathbf{x}_0, r)$. Cauchy's inequality yields

$$|u_t \boldsymbol{\nabla} u \cdot \mathbf{n}| \leqslant \frac{1}{2}\left[u_t^2 + |\boldsymbol{\nabla}_{\mathbf{n}} u|^2\right] \leqslant \frac{1}{2}\left[u_t^2 + |\boldsymbol{\nabla} u|^2\right],$$

where the final inequality follows from the fact that $\mathbf{n}$ is a unit vector. Since $\Box u = 0$, it therefore follows from the above formula that

$$\frac{d}{dt}E(t) \leqslant 0.$$

Hence $E(t) \leqslant E(0) = 0$. Since $E(t) \geqslant 0$, it follows that $E(t) = 0$ for $0 \leqslant t \leqslant t_0$. Again, this implies that $u_t = \boldsymbol{\nabla} u = 0$, so u is constant. Therefore $u = 0$ as required. □

Remark 1.5. Energy estimate techniques can be applied to considerably more general hyperbolic partial differential equations. See, e.g., [4] for more information on this and other techniques.

1.3. **Spherical means and $n \geqslant 2$.** Suppose we wish to solve

$$\Box u = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{1.7a}$$

$$u = g, \quad u_t = h \qquad \text{on } \{0\} \times \mathbb{R}^n. \tag{1.7b}$$

Plan. We wish to derive explicit formulae for $u(t, \mathbf{x})$ in terms of g, h. The plan is to average u over spheres. The averages of u then satisfy the Euler–Poisson–Darboux equation which, for n odd, we can solve using the one-dimensional d'Alembert formula. We then recover u from the limit of its average over smaller and smaller spheres.

Averages. Let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Let $|S(\mathbf{x}, r)|$ and $|B(\mathbf{x}, r)|$ denote the $(n-1)$ and n-dimensional volume of the $(n-1)$-dimensional sphere $S(\mathbf{x}, r) \subset \mathbb{R}^n$ and the n-dimensional ball $B(\mathbf{x}, r) \subset \mathbb{R}^n$, respectively. Denoting the $(n-1)$-volume of the unit sphere S^{n-1} in $\mathbb{R}^n$ by ω_{n-1}, we therefore have

$$|S(\mathbf{x}, r)| = \omega_{n-1} r^{n-1}.$$

We note that

$$|B(\mathbf{x}, r)| = \int_0^r |S(\mathbf{x}, s)|\, ds = \frac{\omega_{n-1}}{n} r^n = \frac{r}{n}|S(\mathbf{x}, r)|.$$

Given a measurable function f on $\mathbb{R}^n$, and a measurable set $E \subseteq \mathbb{R}^n$, we define the average

$$⨍_E f := \frac{1}{|E|}\int_E f.$$

We will be particularly interested in the averages over balls and spheres, i.e. $⨍_{B(\mathbf{x},r)} f$ and $⨍_{S(\mathbf{x},r)} f$.

We now return to the wave equation (1.7). Let $\mathbf{x} \in \mathbb{R}^n$, $t > 0$ and $r > 0$. We define the averaged quantities

$$U(\mathbf{x}, t, r) := ⨍_{S(\mathbf{x},r)} u(t, \mathbf{y})\, dS(\mathbf{y}),$$

$$G(\mathbf{x}, r) := ⨍_{S(\mathbf{x},r)} g(\mathbf{y})\, dS(\mathbf{y}),$$

$$H(\mathbf{x}, r) := ⨍_{S(\mathbf{x},r)} h(\mathbf{y})\, dS(\mathbf{y}).$$

Remark 1.6. The aim is to show that $U(\mathbf{x}, t, r)$ satisfies a wave equation in (t, r) space, which we can solve using the d'Alembert formula. Given $U(\mathbf{x}, t, r)$ for $r > 0$, we then recover $u(t, \mathbf{x})$ as $\lim_{r\to 0+} U(\mathbf{x}, t, r)$.

Proposition 1.7. *Let u satisfy (1.7) and $\mathbf{x} \in \mathbb{R}^n$ be given. Then U satisfies*

$$\partial_t^2 U(\mathbf{x}, t, r) - \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r U(\mathbf{x}, t, r)\right) = 0 \qquad \text{in } (0,\infty) \times (0,\infty), \tag{1.8a}$$

$$U = G, \quad \partial_t U = H \qquad \text{on } \{0\} \times (0,\infty). \tag{1.8b}$$

Proof. Let $d\omega$ denote the volume element on the unit sphere in $\mathbb{R}^n$. We then have, for $\epsilon > 0$,

$$\begin{aligned}
\frac{U(\mathbf{x},t,r+\epsilon)-U(\mathbf{x},t,r)}{\epsilon} &= \frac{1}{\epsilon}\left(\frac{1}{S(\mathbf{x},r+\epsilon)}\int_{S(\mathbf{x},r+\epsilon)} u(t,\mathbf{y})\,dS(\mathbf{y}) - \frac{1}{S(\mathbf{x},r)}\int_{S(\mathbf{x},r)} u(t,\mathbf{y})\,dS(\mathbf{y})\right)\\
&= \frac{1}{\omega_{n-1}\epsilon}\left(\int_{S^{n-1}} u(t,\mathbf{x}+(r+\epsilon)\mathbf{y})\,d\omega(\mathbf{y}) - \int_{S^{n-1}} u(t,\mathbf{x}+r\mathbf{y})\,d\omega(\mathbf{y})\right)\\
&= \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \frac{1}{\epsilon}\left[u(t,\mathbf{x}+(r+\epsilon)\mathbf{y}) - u(t,\mathbf{x}+r\mathbf{y})\right] d\omega(\mathbf{y})\\
&\to \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathbf{n}\cdot\boldsymbol{\nabla} u(t,\mathbf{x}+r\mathbf{y})\,d\omega(\mathbf{y}) \qquad \text{as } \epsilon\to 0,
\end{aligned}$$

where $\mathbf{n}$ denotes the unit normal to the sphere $S(\mathbf{x},r)\subset\mathbb{R}^n$. We therefore have

$$\begin{aligned}
\partial_r U(\mathbf{x},t,r) &= \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathbf{n}\cdot\boldsymbol{\nabla} u(t,\mathbf{x}+r\mathbf{y})\,d\omega(\mathbf{y})\\
&= \frac{1}{|S(\mathbf{x},r)|}\int_{S(\mathbf{x},r)} \mathbf{n}\cdot\boldsymbol{\nabla} u(t,\mathbf{y})\,dS(\mathbf{y}) = \frac{1}{|S(\mathbf{x},r)|}\int_{\partial B(\mathbf{x},r)} \mathbf{n}\cdot\boldsymbol{\nabla} u(t,\mathbf{y})\,dS(\mathbf{y})\\
&= \frac{1}{|S(\mathbf{x},r)|}\int_{B(\mathbf{x},r)} \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} u(t,\mathbf{y})\,d\mathbf{y} = \frac{r}{n|B(\mathbf{x},r)|}\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\\
&= \frac{r}{n}⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}
\end{aligned}$$

Differentiating, and using the fact that $\partial_r|B(\mathbf{x},r)| = |S(\mathbf{x},r)|$, we have

$$\begin{aligned}
\partial_r^2 U(\mathbf{x},t,r) &= \frac{1}{n}⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y} + \frac{r}{n}\partial_r\left(\frac{1}{|B(\mathbf{x},r)|}\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\right)\\
&= \frac{1}{n}⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y} - \frac{r}{n}\frac{|S(x,r)|}{|B(\mathbf{x},r)|^2}\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\\
&\qquad\qquad + \frac{r}{n}\frac{1}{|B(\mathbf{x},r)|}\partial_r\left(\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\right)\\
&= \left(\frac{1}{n}-1\right)⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y} + \frac{1}{|S(\mathbf{x},r)|}\partial_r\left(\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\right).
\end{aligned}$$

We also have

$$\partial_r\left(\int_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\right) = \partial_r\left(\int_0^r\int_{S(\mathbf{x},s)} \Delta u(t,\mathbf{y})\,dS\,ds\right) = \int_{S(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,dS(\mathbf{y})$$

and therefore

$$\partial_r^2 U(\mathbf{x},t,r) = \left(\frac{1}{n}-1\right)⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y} + ⨍_{S(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,dS(\mathbf{y}).$$

We now have

$$\begin{aligned}
\partial_t^2 U(\mathbf{x},t,r) &= ⨍_{S(\mathbf{x},r)} u_{tt}(t,\mathbf{y})\,dS(\mathbf{y}) = ⨍_{S(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,dS(\mathbf{y})\\
&= \partial_r^2 U(\mathbf{x},t,r) - \left(\frac{1}{n}-1\right)⨍_{B(\mathbf{x},r)} \Delta u(t,\mathbf{y})\,d\mathbf{y}\\
&= \partial_r^2 U(\mathbf{x},t,r) - \left(\frac{1}{n}-1\right)\frac{n}{r}\partial_r U(\mathbf{x},t,r)\\
&= \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r U(\mathbf{x},t,r)\right),
\end{aligned}$$

as required. In order to derive the boundary conditions, we note that

$$U(\mathbf{x}, 0, r) = ⨍_{S(\mathbf{x},r)} u(0, \mathbf{y})\, dS(\mathbf{y}) = ⨍_{S(\mathbf{x},r)} g(\mathbf{y})\, dS(\mathbf{y}) = G(\mathbf{x}, r),$$

and similarly for $\partial_t U(\mathbf{x}, 0, r)$. □

Wave equation for $n = 3$. For $n = 3$, the average $U(\mathbf{x}, t, r)$ satisfies

$$\partial_t^2 U(\mathbf{x}, t, r) = \frac{1}{r^2} \partial_r \left(r^2 \partial_r U(\mathbf{x}, t, r) \right).$$

Letting $U(\mathbf{x}, t, r) = \widetilde{U}(\mathbf{x}, t, r)/r$, we find that

$$\partial_t^2 \widetilde{U}(\mathbf{x}, t, r) = \partial_r^2 \widetilde{U}(\mathbf{x}, t, r).$$

In addition,

$$\widetilde{U}(\mathbf{x}, 0, r) = rU(\mathbf{x}, 0, r) = rG(\mathbf{x}, r) =: \widetilde{G}(\mathbf{x}, r),$$
$$\partial_t \widetilde{U}(\mathbf{x}, 0, r) = r\partial_t U(\mathbf{x}, 0, r) = rH(\mathbf{x}, r) =: \widetilde{H}(\mathbf{x}, r).$$

It follows from the d'Alembert formula that, for $t > r$, we have

$$\widetilde{U}(\mathbf{x}, t, r) = \frac{1}{2} \left(\widetilde{G}(\mathbf{x}, t + r) - \widetilde{G}(\mathbf{x}, t - r) \right) + \frac{1}{2} \int_{t-r}^{t+r} \widetilde{H}(\mathbf{x}, u)\, ds.$$

We therefore have

$$\begin{aligned}
u(t, \mathbf{x}) &= \lim_{r \to 0+} U(\mathbf{x}, t, r) = \lim_{r \to 0+} \frac{\widetilde{U}(\mathbf{x}, t, r)}{r} \\
&= \lim_{r \to 0+} \left[\frac{1}{2r} \left(\widetilde{G}(\mathbf{x}, t + r) - \widetilde{G}(\mathbf{x}, t - r) \right) + \frac{1}{2r} \int_{t-r}^{t+r} \widetilde{H}(\mathbf{x}, u)\, ds \right] \\
&= \widetilde{G}'(\mathbf{x}, t) + \widetilde{H}(\mathbf{x}, t) \\
&= \partial_t \left(tG(\mathbf{x}, t) \right) + tH(\mathbf{x}, t) \\
&= t\partial_t G(\mathbf{x}, t) + G(\mathbf{x}, t) + tH(\mathbf{x}, t) \\
&= t\partial_t G(\mathbf{x}, t) + ⨍_{S(\mathbf{x},t)} \left[th(\mathbf{y}) + g(\mathbf{y}) \right] dS(\mathbf{y}).
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
t\partial_t G(\mathbf{x}, t) &= t\partial_t \left[\frac{1}{|S(\mathbf{x}, t)|} \int_{S(\mathbf{x},t)} g(\mathbf{y})\, dS(\mathbf{y}) \right] \\
&= t \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\frac{1}{|S(\mathbf{x}, t + \epsilon)|} \int_{S(\mathbf{x},t+\epsilon)} g(\mathbf{y})\, dS(\mathbf{y}) - \frac{1}{|S(\mathbf{x}, t)|} \int_{S(\mathbf{x},t)} g(\mathbf{y})\, dS(\mathbf{y}) \right] \\
&= \frac{t}{\omega_{n-1}} \lim_{\epsilon \to 0} \int_{S^{n-1}} \frac{g(\mathbf{x} + (t + \epsilon)\mathbf{y}) - g(\mathbf{x} + t\mathbf{y})}{\epsilon}\, d\omega(\mathbf{y}) \\
&= \frac{t}{\omega_{n-1}} \int_{S^{n-1}} \langle \mathbf{n}, \boldsymbol{\nabla} g(\mathbf{x} + t\mathbf{y}) \rangle\, d\omega(\mathbf{y}) \\
&= t ⨍_{S(\mathbf{x},t)} \left\langle \frac{\mathbf{y} - \mathbf{x}}{t}, \boldsymbol{\nabla} g(\mathbf{y}) \right\rangle dS(\mathbf{y}) \\
&= ⨍_{S(\mathbf{x},t)} \langle \mathbf{y} - \mathbf{x}, \boldsymbol{\nabla} g(\mathbf{y}) \rangle\, dS(\mathbf{y})
\end{aligned}$$

We have therefore derived the *Kirchoff formula*:

$$u(t, \mathbf{x}) = ⨍_{S(\mathbf{x},t)} \left[th(\mathbf{y}) + g(\mathbf{y}) + \langle \mathbf{y} - \mathbf{x}, \boldsymbol{\nabla} g(\mathbf{y}) \rangle \right] dS(\mathbf{y})$$

Remarks 1.8. (1) Unlike in one-dimension, the formula for $u(t,\mathbf{x})$ depends on the derivative of g. In higher dimensions, we will require higher derivatives of g. As such, regularity of g plays a more significant role in higher dimensions.

(2) The value of $u(t,\mathbf{x})$ only depends on the values of g, h and $\boldsymbol{\nabla} g$ on the sphere $S(\mathbf{x},t)$.

Wave equation for $n = 2$. The method of descent. The above construction cannot be used in the case $n = 2$. We proceed rather by embedding the two-dimensional problem into the three-dimensional problem and using the $3D$ result. Given $\mathbf{x} = (x^1, x^2) \in \mathbb{R}^2$, let $\overline{\mathbf{x}} = (x^1, x^2, x^3) \in \mathbb{R}^3$ be a point in $\mathbb{R}^3$ that projects to $\mathbf{x}$ under projection to the (x^1, x^2) plane. We want $u(t,\mathbf{x})$ to be a solution of the problem

$$\Box u = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^2, \tag{1.9a}$$

$$u = g, \quad u_t = h \qquad \text{on } \{0\}\times\mathbb{R}^2. \tag{1.9b}$$

we define corresponding quantities on $\mathbb{R}^3$ by trivially extending in the x^3 direction, i.e.

$$\overline{u}(t,\overline{\mathbf{x}}) := u(t,\mathbf{x}), \qquad \overline{g}(\overline{\mathbf{x}}) := g(\mathbf{x}), \qquad \overline{h}(\overline{\mathbf{x}}) := h(\mathbf{x}).$$

It then follows that $\overline{u}$, etc, satisfy the three-dimensional conditions

$$\overline{\Box}\,\overline{u} = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^3, \tag{1.10a}$$

$$\overline{u}(0,\overline{\mathbf{x}}) = \overline{g}(\overline{\mathbf{x}}), \quad \overline{u}_t(t,\overline{\mathbf{x}}) = \overline{h}(\overline{\mathbf{x}}) \qquad \overline{\mathbf{x}} \in \mathbb{R}^3, \tag{1.10b}$$

where $\overline{\Box}$ and $\overline{\Delta}$ denote the $n = 3$ wave operator and Laplacian, respectively. From the discussion in the previous section, we therefore deduce that

$$u(t,\mathbf{x}) = \overline{u}(t,\overline{\mathbf{x}}) = \partial_t\left(t \fint_{\overline{S}(\overline{\mathbf{x}},t)} \overline{g}(\overline{\mathbf{y}})\, d\overline{S}(\overline{\mathbf{y}})\right) + t\fint_{\overline{S}(\overline{\mathbf{x}},t)} \overline{h}(\overline{\mathbf{y}})\, d\overline{S}(\overline{\mathbf{y}}), \tag{1.11}$$

where

$$\overline{S}(\overline{\mathbf{x}},t) := \left\{\overline{\mathbf{y}} \in \mathbb{R}^3 \,\middle|\, |\overline{\mathbf{y}} - \overline{\mathbf{x}}|_{\mathbb{R}^3} = t\right\} \equiv \left\{\overline{\mathbf{y}} \in \mathbb{R}^3 \,\middle|\, y^3 = \pm\sqrt{t^2 - |\mathbf{y}-\mathbf{x}|^2_{\mathbb{R}^2}}\right\}$$

denotes the two-sphere in $\mathbb{R}^3$ with centre $\overline{\mathbf{x}}$ and radius t, and $d\overline{S}$ denotes the natural area element on $\overline{S}(\overline{\mathbf{x}},t)$. We therefore have

$$\begin{aligned}
\fint_{\overline{S}(\overline{\mathbf{x}},t)} \overline{g}(\overline{\mathbf{y}})\, d\overline{S}(\overline{\mathbf{y}}) &= \frac{1}{4\pi t^2}\int_{\overline{S}(\overline{\mathbf{x}},t)} g(y^1,y^2)\, d\overline{S}(y^1,y^2,y^3) \\
&= \frac{1}{4\pi t^2}\int_{|\mathbf{y}-\mathbf{x}|=0}^{t}\left(\int_{y^3=\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2_{\mathbb{R}^2}}} + \int_{y^3=-\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2_{\mathbb{R}^2}}}\right) g(\mathbf{y})\, d\overline{S}(\mathbf{y},y^3) \\
&= \frac{2}{4\pi t^2}\int_{|\mathbf{y}-\mathbf{x}|=0}^{t}\int_{y^3=\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2_{\mathbb{R}^2}}} g(\mathbf{y})\, d\overline{S}(\mathbf{y},y^3)
\end{aligned}$$

For simplicity, letting $\mathbf{x} = 0$ (alternatively let $\mathbf{z} := \mathbf{y} - \mathbf{x}$ and work with $\mathbf{z}$ rather than $\mathbf{y}$), the induced metric on the set $y^3 = \sqrt{t^2 - |\mathbf{y}|^2_{\mathbb{R}^2}}$ is

$$\begin{aligned}
|d\mathbf{y}|^2 + (dy^3)^2 = |d\mathbf{y}|^2 + \left(\frac{1}{2\sqrt{t^2-|\mathbf{y}|^2_{\mathbb{R}^2}}} - 2\mathbf{y}\cdot d\mathbf{y}\right)^2 &= |d\mathbf{y}|^2 + \frac{1}{t^2-|\mathbf{y}|^2_{\mathbb{R}^2}}\,(\mathbf{y}\cdot d\mathbf{y})^2 \\
= \left(1 + \frac{(y^1)^2}{t^2-(y^1)^2-(y^2)^2}\right)(dy^1)^2 &+ \left(1 + \frac{(y^2)^2}{t^2-(y^1)^2-(y^2)^2}\right)(dy^2)^2 \\
&+ \frac{2y^1y^2}{t^2-(y^1)^2-(y^2)^2}dy^1dy^2.
\end{aligned}$$

It follows that

$$d\overline{S}(\mathbf{y},y^3) = \left(1 + \frac{|\mathbf{y}|^2}{t^2-|\mathbf{y}|^2}\right)^{1/2} d\mathbf{y}.$$

Putting the $\mathbf{x}$ dependence back in, we have

$$d\overline{S}(\mathbf{y}, y^3) = \left(1 + \frac{|\mathbf{y}-\mathbf{x}|^2}{t^2 - |\mathbf{y}-\mathbf{x}|^2}\right)^{1/2} d\mathbf{y}.$$

Therefore,

$$\begin{aligned}
⨍_{\overline{S}(\overline{\mathbf{x}},t)} \overline{g}(\overline{\mathbf{y}})\, d\overline{S}(\overline{\mathbf{y}}) &= \frac{1}{2\pi t^2} \int_{|\mathbf{y}-\mathbf{x}|=0}^{t} g(\mathbf{y}) \left(1 + \frac{|\mathbf{y}-\mathbf{x}|^2}{t^2 - |\mathbf{y}-\mathbf{x}|^2}\right)^{1/2} d\mathbf{y} \\
&= \frac{1}{2\pi t} \int_{B(\mathbf{x},t)} g(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}}, \\
&= \frac{t}{2} ⨍_{B(\mathbf{x},t)} g(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}},
\end{aligned}$$

where $B(\mathbf{x}, t)$ now again denotes the ball in $\mathbb{R}^2$. From (1.11), we therefore deduce that

$$u(t, \mathbf{x}) = \partial_t \left(\frac{t^2}{2} ⨍_{B(\mathbf{x},t)} g(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}} \right) + \frac{t^2}{2} ⨍_{B(\mathbf{x},t)} h(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}}.$$

Finally, we have

$$\begin{aligned}
\partial_t \left(\frac{t^2}{2} ⨍_{B(\mathbf{x},t)} g(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}} \right) &= \partial_t \left(\frac{t^2}{2} \frac{1}{|B(\mathbf{x},t)|} \int_{B(\mathbf{x},t)} g(\mathbf{y}) \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}} \right) \\
&= \partial_t \left(\frac{1}{2\pi} \int_{B(0,1)} g(\mathbf{x} + t\mathbf{z}) \frac{t d\mathbf{z}}{\left(1 - |\mathbf{z}|^2\right)^{1/2}} \right) \\
&= \frac{1}{2\pi} \int_{B(0,1)} \left[g(\mathbf{x} + t\mathbf{z}) + t\langle \mathbf{z}, \boldsymbol{\nabla} g(\mathbf{x} + t\mathbf{z})\rangle \right] \frac{d\mathbf{z}}{\left(1 - |\mathbf{z}|^2\right)^{1/2}} \\
&= \frac{1}{2\pi} \int_{B(\mathbf{x},t)} \left[g(\mathbf{y}) + \langle \mathbf{y} - \mathbf{x}, \boldsymbol{\nabla} g(\mathbf{y})\rangle \right] \frac{d\mathbf{y}}{t\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}} \\
&= \frac{t}{2} ⨍_{B(\mathbf{x},t)} \left[g(\mathbf{y}) + \langle \mathbf{y} - \mathbf{x}, \boldsymbol{\nabla} g(\mathbf{y})\rangle \right] \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}}
\end{aligned}$$

We therefore have the *Poisson formula* for the solution of the two-dimensional wave equation

$$u(t, \mathbf{x}) = \frac{1}{2} ⨍_{B(\mathbf{x},t)} \left[t g(\mathbf{y}) + t\langle \mathbf{y} - \mathbf{x}, \boldsymbol{\nabla} g(\mathbf{y})\rangle + t^2 h(\mathbf{y}) \right] \frac{d\mathbf{y}}{\left(t^2 - |\mathbf{y}-\mathbf{x}|^2\right)^{1/2}}.$$

Remark 1.9. In two dimensions, $u(t, \mathbf{x})$ depends on the data g, h and $\boldsymbol{\nabla} g$ on the whole of the ball $B(\mathbf{x}, t)$, rather than just on the boundary $S(\mathbf{x}, t)$ which was the case for $n = 3$.

2. Fourier methods

An alternative method of constructing solutions of the wave equation is by means of Fourier transforms.[2] Given $u \in L^1(\mathbb{R}^n)$, we define the Fourier transform

$$\hat{u}(\boldsymbol{\xi}) \equiv (\mathcal{F}u)(\boldsymbol{\xi}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}.$$

We list some properties of $\hat{u}$:

(1) $\hat{u} \in L^\infty(\mathbb{R}^n)$, since $|\hat{u}(\boldsymbol{\xi})| \leqslant \frac{1}{(2\pi)^{n/2}} \|u\|_{L^1(\mathbb{R}^n)} < \infty$.
(2) $\hat{u}$ is continuous. (This follows from the dominated convergence theorem.)
(3) *The Riemann–Lebesgue lemma*: $\hat{u}(\boldsymbol{\xi}) \to 0$ as $|\boldsymbol{\xi}| \to \infty$.

It is convenient to introduce a space of functions that is closed under Fourier transform.[3] Given multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, with $\alpha_1, \ldots, \beta_n$ non-negative integers, we define

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad D^\beta := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Given a smooth function $u \in C^\infty(\mathbb{R}^n)$ and multi-indices α, β, we define the semi-norm

$$p_{\alpha,\beta}(u) := \sup_{\mathbf{x}\in\mathbb{R}^n} |x^\alpha D^\beta u(\mathbf{x})|.$$

Letting $|\alpha| := \alpha_1 + \cdots + \alpha_n$, etc, we define the increasing family of semi-norms

$$Q_k(u) := \sum_{\alpha,\beta\leqslant k} p_{\alpha,\beta}(u).$$

Definition 2.1. The *Schwartz space of rapidly decreasing functions* is defined to be

$$\mathscr{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n)\, | p_{\alpha,\beta}(u) < \infty, \text{for all multi-indices } \alpha, \beta\}.$$

We also define the space of *slowly increasing smooth functions*

$$\mathcal{O}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall\alpha, \exists N \in \mathbb{N}_0, \exists C \geqslant 0 \text{ s.t. } \forall \mathbf{x} \in \mathbb{R}^n, |D^\alpha f(\mathbf{x})| \leqslant C\,(1+|\mathbf{x}|)^N \right\}$$

Remarks 2.2.

(1) If $u \in \mathscr{S}(\mathbb{R}^n)$, then u and all of its derivatives fall off faster than any inverse power of x as $|\mathbf{x}| \to \infty$. In particular, elements of u lie in $L^1(\mathbb{R}^n)$ and have a well-defined Fourier transform.
(2) If $u \in \mathscr{S}(\mathbb{R}^n)$ and $f \in \mathcal{O}(\mathbb{R}^n)$ then $fu \in \mathscr{S}(\mathbb{R}^n)$.

Proposition 2.3. *$u \in \mathscr{S}(\mathbb{R}^n)$ then $\hat{u} \in \mathscr{S}(\mathbb{R}^n)$, i.e. $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.*

Proof.

$$\xi^\alpha D_\xi^\beta \hat{u}(\boldsymbol{\xi}) = \text{const.} \times \int D_x^\alpha \left(x^\beta u(x)\right) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x}.$$

Therefore

$$\left|\xi^\alpha D_\xi^\beta \hat{u}(\boldsymbol{\xi})\right| \lesssim \int \left|D_x^\alpha \left(x^\beta u(x)\right)\right| d\mathbf{x} < \infty,$$

since the integrand is rapidly decreasing. □

We also define the operator

$$(\mathcal{F}^* u)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\boldsymbol{\xi}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}.$$

By the same argument as in Proposition 2.3, we deduce that $\mathcal{F}^*\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

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[2] Most of the material in this section is derived from Chapters 3.3–3.5 of [5]. For more information on Schwartz spaces and tempered distributions see, for instance, Chapter 5 of [2].

[3] Note that, given $u \in L^1(\mathbb{R}^n)$, in general $\hat{u} \notin L^1(\mathbb{R}^n)$, so the L^1 property is not preserved under Fourier transform.

Theorem 2.4 (The Fourier inversion formula)**.**

$$\mathcal{F} \circ \mathcal{F}^* = \mathcal{F}^* \circ \mathcal{F} = \mathrm{Id}$$

on $\mathscr{S}(\mathbb{R}^n)$, i.e. the operators $\mathcal{F}$ and $\mathcal{F}^*$ are inverses on $\mathscr{S}(\mathbb{R}^n)$.

We will also be interested in the dual space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions, i.e. continuous linear maps $\mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$.[4] More specifically:

Definition 2.5. A linear map $\sigma\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ lies in $\mathscr{S}'(\mathbb{R}^n)$ if there exists $C \geqslant 0$ and $N \in \mathbb{N}_0$ such that

$$|\langle \sigma, u \rangle| \leqslant C Q_N(u), \qquad \forall u \in \mathscr{S}(\mathbb{R}^n).$$

Remark 2.6. Given a sequence (u_k) in $\mathscr{S}(\mathbb{R}^n)$, we say that $u_k \to 0$ if $p_{\alpha,\beta}(u) \to 0$ for all α, β. Definition 2.5 is then equivalent to stating that a linear map $\sigma\colon \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ lies in $\mathscr{S}'(\mathbb{R}^n)$ if for all sequences (u_k) in $\mathscr{S}(\mathbb{R}^n)$ with $u_k \to 0$ we have $\langle \sigma, u_k \rangle \to 0$ as $k \to \infty$.

We define the action of $\mathcal{F}$ and $\mathcal{F}^*$ on $\mathscr{S}'(\mathbb{R}^n)$ by duality, so

$$\langle \mathcal{F}\sigma, u \rangle := \langle \sigma, \mathcal{F}u \rangle, \qquad \forall u \in \mathscr{S}(\mathbb{R}^n),$$

and similarly for $\mathcal{F}^*$. The inversion formula on $\mathscr{S}(\mathbb{R}^n)$ then implies that $\mathcal{F} \circ \mathcal{F}^* = \mathcal{F}^* \circ \mathcal{F} = \mathrm{Id}$ also holds on $\mathscr{S}'(\mathbb{R}^n)$.

We consider the wave equation $\Box u(t, \mathbf{x}) = 0$ on $(0, \infty) \times \mathbb{R}^n$.

The point is that $\hat{u}$ satisfies the Fourier transform of the wave equation:

$$\frac{\partial^2}{\partial t^2} \hat{u}(t, \boldsymbol{\xi}) + |\boldsymbol{\xi}|^2 \hat{u}(t, \boldsymbol{\xi}) = 0. \tag{2.1}$$

Proof.

$$\begin{aligned}
\frac{\partial^2 \hat{u}(t, \boldsymbol{\xi})}{\partial t^2} &= \frac{\partial^2}{\partial t^2} \frac{1}{(2\pi)^{n/2}} \int u(t,x) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \frac{\partial^2 u(t,x)}{\partial t^2} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x} \qquad \text{(because } u \text{ rapidly decreasing)} \\
&= \frac{1}{(2\pi)^{n/2}} \int \Delta u(t,x) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int u(t,x) \Delta e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x} \qquad \text{(integration by parts and } u \text{ rapidly decreasing)} \\
&= -\frac{1}{(2\pi)^{n/2}} \int u(t,x) |\boldsymbol{\xi}|^2 e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\mathbf{x} \\
&= -|\boldsymbol{\xi}|^2 \hat{u}(t, \boldsymbol{\xi}).
\end{aligned}$$

□

We assume the initial conditions

$$\hat{u}(0, \boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi}), \qquad \partial_t \hat{u}(0, \boldsymbol{\xi}) = \hat{h}(\boldsymbol{\xi}).$$

The corresponding solution of (2.1) may then be written as

$$\hat{u}(t, \boldsymbol{\xi}) = \hat{h}(\boldsymbol{\xi}) \frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|} + \hat{g}(\boldsymbol{\xi}) \cos(|\boldsymbol{\xi}|t). \tag{2.2}$$

Note that $\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|}$ and $\cos(|\boldsymbol{\xi}|t)$ are slowly increasing smooth functions of $\boldsymbol{\xi}$, i.e. elements of $\mathcal{O}(\mathbb{R}^n)$. Given $F \in \mathcal{O}(\mathbb{R}^n)$ and $\sigma \in \mathscr{S}'(\mathbb{R}^n)$, then $F\sigma \in \mathscr{S}'(\mathbb{R}^n)$, where multiplication is defined by the relation

$$\langle F\sigma, u \rangle := \langle \sigma, Fu \rangle, \qquad \forall u \in \mathscr{S}(\mathbb{R}^n).$$

Given $\hat{g}, \hat{h} \in \mathscr{S}'(\mathbb{R}^n)$, it follows that the $\hat{u}(t, \cdot)$ defined in equation (2.2) is a well-defined element of $\mathscr{S}'(\mathbb{R}^n)$ for all $t > 0$. The inversion formula then gives $u(t, \cdot) \in \mathscr{S}'(\mathbb{R}^n)$ for $t > 0$

[4] It is conventional to work with complex-valued functions.

Example 2.7. If $g, h \in \mathscr{S}(\mathbb{R}^n)$, then $\hat{g}, \hat{h} \in \mathscr{S}(\mathbb{R}^n)$. It follows that the functions $\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|}\hat{h}(\boldsymbol{\xi})$ and $\cos(|\boldsymbol{\xi}|t)\,\hat{g}(\boldsymbol{\xi})$ lie in $\mathscr{S}(\mathbb{R}^n)$, for all $t > 0$, and hence that $\hat{u}(t,\cdot) \in \mathscr{S}(\mathbb{R}^n)$ for $t > 0$. Therefore, $u(t,\cdot) \in \mathscr{S}(\mathbb{R}^n)$ for all $t > 0$.

Example 2.8. Take $g = 0$, $h = \delta \in \mathscr{S}'(\mathbb{R}^n)$. It then follows that $\hat{g} = 0$ and $\hat{h} = 1 \in \mathscr{S}'(\mathbb{R}^n)$. The solution of the corresponding initial value problem is called the *fundamental solution of the wave equation*

$$\hat{R}(t,\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}}\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|}.$$

2.1. L^2 **estimates.** Finally, we study some L^2 estimates for solutions of the wave-equation using the explicit formula (2.2). If $u \in L^2(\mathbb{R}^n)$ then $\hat{u} \in L^2(\mathbb{R}^n)$ and we have the Parseval identity

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Noting that

$$\left|\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|}\right| \leqslant t, \qquad |\cos(|\boldsymbol{\xi}|t)| \leqslant 1,$$

we deduce from (2.2) that

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} &= \|\hat{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\
&= \left\|\hat{h}(\boldsymbol{\xi})\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|} + \hat{g}(\boldsymbol{\xi})\cos(|\boldsymbol{\xi}|t)\right\|_{L^2_{\boldsymbol{\xi}}(\mathbb{R}^n)} \\
&\leqslant \left\|\hat{h}(\boldsymbol{\xi})\frac{\sin(|\boldsymbol{\xi}|t)}{|\boldsymbol{\xi}|}\right\|_{L^2_{\boldsymbol{\xi}}(\mathbb{R}^n)} + \|\hat{g}(\boldsymbol{\xi})\cos(|\boldsymbol{\xi}|t)\|_{L^2_{\boldsymbol{\xi}}(\mathbb{R}^n)} \\
&\leqslant t\left\|\hat{h}\right\|_{L^2(\mathbb{R}^n)} + \|\hat{g}\|_{L^2(\mathbb{R}^n)} \\
&= t\|h\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Similarly,

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant \|h\|_{L^2(\mathbb{R}^n)} + \left\||\boldsymbol{\xi}|\,\hat{g}(\boldsymbol{\xi})\right\|_{L^2_{\boldsymbol{\xi}}(\mathbb{R}^n)}.$$

Note that $(\mathcal{F}(\boldsymbol{\nabla} f))(\boldsymbol{\xi}) = i\boldsymbol{\xi}\hat{f}(\boldsymbol{\xi})$. Therefore

$$\|\boldsymbol{\nabla} f\|^2_{L^2_{\mathbf{x}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\boldsymbol{\nabla} f(\mathbf{x})|^2\,d\mathbf{x} = \int_{\mathbb{R}^n} |\mathcal{F}(\boldsymbol{\nabla} f)(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = \int_{\mathbb{R}^n} |\xi|^2\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} = \left\||\boldsymbol{\xi}|\hat{f}(\boldsymbol{\xi})\right\|^2_{L^2_{\boldsymbol{\xi}}(\mathbb{R}^n)}$$

We therefore deduce that

$$\|u_t(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant \|h\|_{L^2(\mathbb{R}^n)} + \|\boldsymbol{\nabla} g\|_{L^2(\mathbb{R}^n)}.$$

In particular, it is sufficient that $f, g \in L^2(\mathbb{R}^n)$ (and, hence, $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$) in order to define $\hat{u}$. For $\partial_t \hat{u}$, it is sufficient that $h \in L^2(\mathbb{R}^n)$ and $\boldsymbol{\nabla} g \in L^2(\mathbb{R}^n)$. Similarly, we find that

$$\|\hat{u}_{tt}\|_{L^2(\mathbb{R}^n)} \leqslant \left\||\boldsymbol{\xi}|\,\hat{h}\right\|_{L^2(\mathbb{R}^n)} + \left\||\boldsymbol{\xi}|^2\hat{g}\right\|_{L^2(\mathbb{R}^n)}$$

and, therefore,

$$\|u_{tt}\|_{L^2(\mathbb{R}^n)} \leqslant \|\boldsymbol{\nabla} h\|_{L^2(\mathbb{R}^n)} + \|\boldsymbol{\nabla\nabla} g\|_{L^2(\mathbb{R}^n)}.$$

In this approach, it is therefore natural that g, h should be elements of Sobolev spaces: $h \in W^{1,2}(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, $g \in W^{2,2}(\mathbb{R}^n) = H^2(\mathbb{R}^n)$. Recall the following.

Definition 2.9. Let $D \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_0$, $1 \leqslant p < \infty$, then

$$W^{k,p}(D) := \{u \in L^p(D)\,|\text{there exists } D^\alpha u \in L^p(\mathbb{R}^n) \text{ for } 0 \leqslant |\alpha| \leqslant k\},$$

where the derivatives $D^\alpha u$ are defined in the distributional sense. We define the norms

$$\|u\|_{W^{k,p}(D)} := \left(\sum_{\alpha \leqslant k}\int_D |D^\alpha u(\mathbf{x})|^p\,d\mathbf{x}\right)^{1/p}.$$

We let $H^k(D) := W^{k,2}(D)$ and note that $L^p(D) = W^{0,p}(D)$.

Remark 2.10. It can be shown that $W^{k,p}(D)$ are Banach spaces, and $H^k(D)$ are Hilbert spaces.

We note that, if $g \in H^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$, then

$$\int g^2 d\mathbf{x} + \int |\boldsymbol{\nabla} g|^2 d\mathbf{x} + \int |\boldsymbol{\nabla}\boldsymbol{\nabla} g|^2 d\mathbf{x} < \infty, \qquad \int h^2 d\mathbf{x} + \int |\boldsymbol{\nabla} h|^2 d\mathbf{x} < \infty.$$

For $g, h \in \mathscr{S}(\mathbb{R}^n)$, all of the above integrals are finite. As such, the Sobolev solutions are weaker than the solutions in $\mathscr{S}(\mathbb{R}^n)$.

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